Research topic no. 6

The Modelling of Fish Stocks

Diana Harambaş, Bianca Crişan, Codruța Jucan, Horațiu Cătărig

Colegiul Național "Emil Racoviță"

Research topic

In order to model the management of fish stocks, we consider the deterministic Schaefer model

$X_{n+1} = X_n + rX_n\left(1 - \frac{X_n}{M}\right) - C$, X_n — the biomass for year n

where

M — maximum biomass that can live in a certain environment

r — rate of growth

C — amount of biomass fished

Can we model, by modifying the parameters, all the possible situations (extinction, uncontrolled growth of species?

We can also introduce an environmental variable — V_n

$$X_{n+1} = \left[X_n + rX_n\left(1 - \frac{X_n}{M}\right) - C\right] V_n$$

We can ask the same questions as before.

Introduction

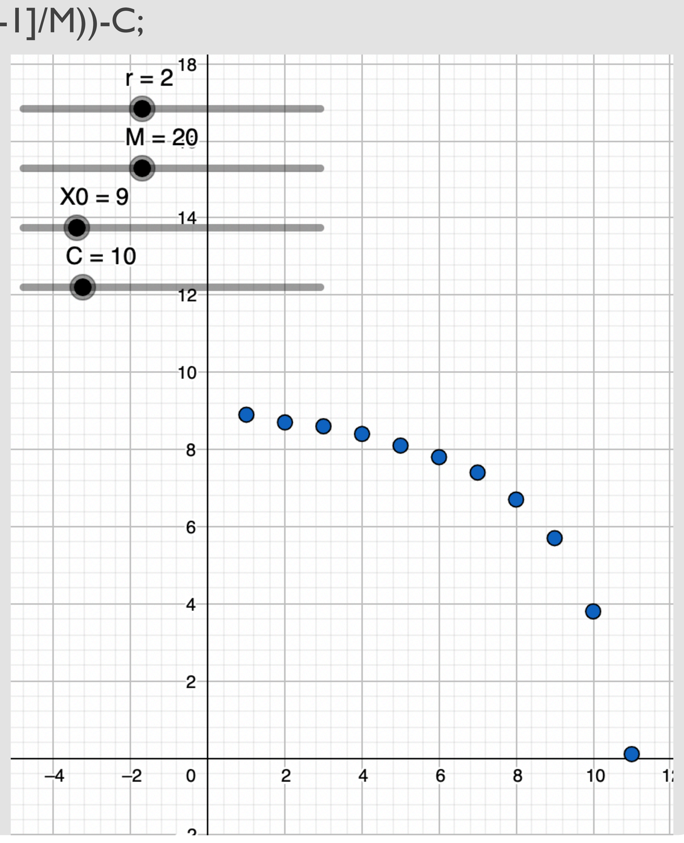
We decided to work on the research topic using differential equations and studying equilibrium points and different particular cases.

Below is a short C++ code that helps us determine values until we reach <u>extinction</u>, for any parameters X_0, M, r, C.

Using more such particular tests and also working with a Geogebra graph tool where we calibrated the parameters, we saw really diverse results. Changing C seemed to produce most of the differences, so we analysed the next cases.

```
void nextgen(int i)
{
   x[i] = x[i-1] + (r*x[i-1]*(1-x[i-1]/M))-C;
}
int main()
{
   cin >> x[0] >> M >> r >> C;

   int i = 0;
   while(x[i]>0)
   {
      i++;
      nextgen(i);
      cout<<" "<<x[i];
   }
   cout<<i;
   return 0;
}
```



Extinction (general case)

If the biomass in one year reaches this certain value, than the next year there will be no more fish:

$$X = \frac{M}{2} + \frac{M}{2}\left[-\sqrt{(r+1)^2 - \frac{4r}{C}} + 1\right]$$

Case $C = 0$

This is the case where no fishing is done and it follows the natural growth of the species. When representing the growth on Geogebra, the graphs converge to M, the maximum biomass.

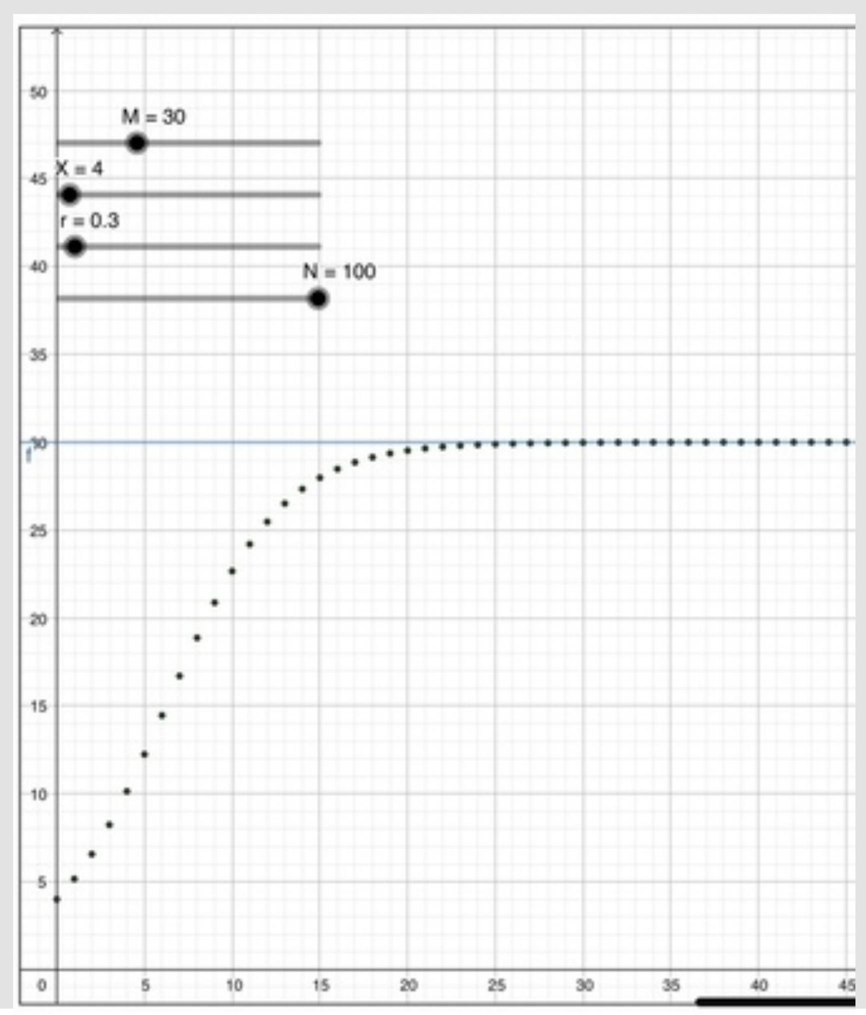
!!! This is not biologically accurate. If there are more fish populating the environment, then there are less resources (food, etc.) !!!

Thus, it's not as probable for the graph to stay the sequence to converge to M and remain constant per the first reach of equilibrium point.

The biomass formula for this case:

$$X_n = \frac{MX_0e^{rn}}{M + X_0(e^{rn} - 1)}$$

A graphic representation for this model of growth:



Case $C \neq 0$

The results depend on the relation between Cr and $\frac{M}{4}$

Case 1: $Cr = \frac{M}{4}$

$$X_n = \frac{M\left[nr\left(Mr - 2X_0\right) - Mr - 2X_0\right]}{2r\left[n\left(Mr - 2X_o\right) - 2M\right]}$$

Case 2: $Cr > \frac{M}{4}$

$$X_n = \frac{1}{2}[M - \sqrt{M(4Cr - M)}\tan\left[r\left(n - \alpha_2\right)\frac{\sqrt{4Cr - M}}{2\sqrt{M}}\right]$$

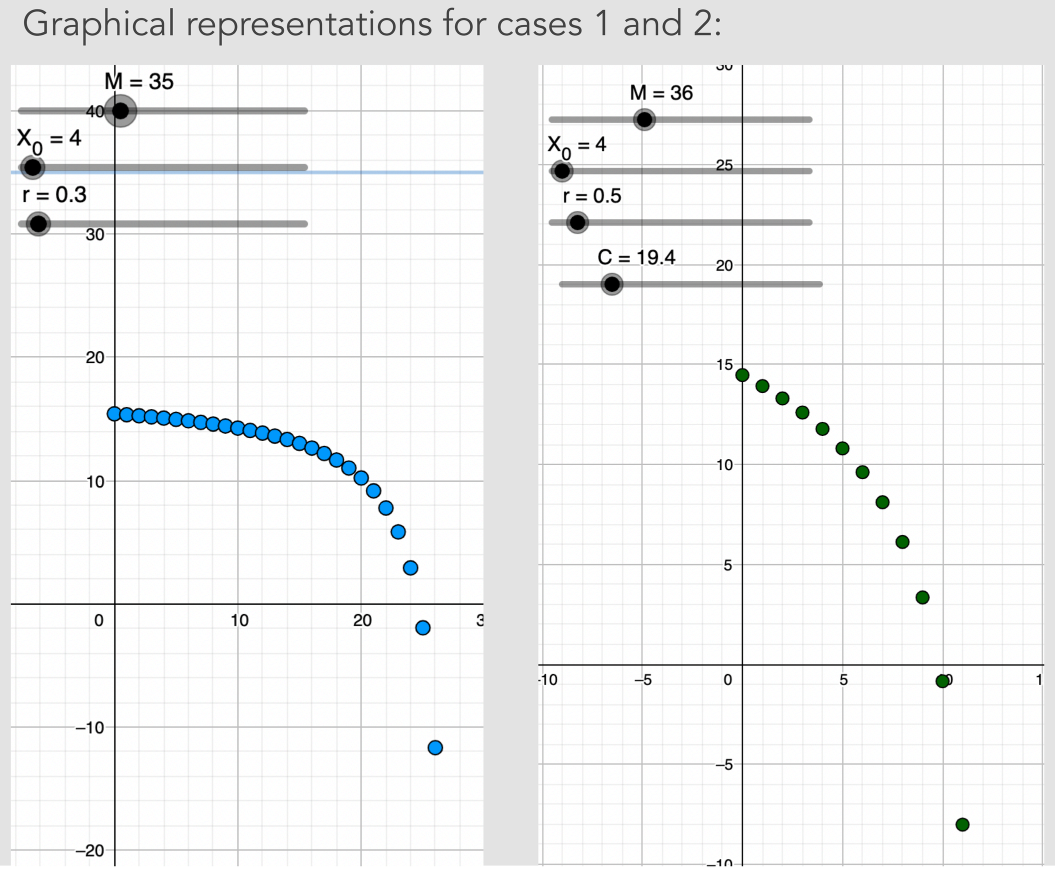
Where $\alpha_2 = \dfrac{2\sqrt{M}\arctan\left(\frac{2X_0 - M}{\sqrt{M(4Cr - M)}}\right)}{\sqrt{4Cr - M}}$

Case 3: $Cr < \frac{M}{4}$

$$X_n = \frac{M}{2} - \sqrt{M\left(\frac{M}{4} - Cr\right)} - \frac{2\sqrt{M\left(\frac{M}{4} - Cr\right)}}{e^{\frac{-2r\sqrt{M\left(\frac{M}{4} - Cr\right)}(n - \alpha_3)}{M}} - 1}$$

Where $\alpha_3 = \dfrac{\ln\left[\frac{X_0 - \frac{M}{2} - \sqrt{M(\frac{M}{4} - Cr)}}{X_0 - \frac{M}{2} + \sqrt{M(\frac{M}{4} - Cr)}}\right]\sqrt{M}}{2r\sqrt{\frac{M}{4} - Cr}}$

Graphical representations for cases 1 and 2:



Conclusions

$\frac{dx}{dn} = rx(1 - \frac{x}{M}) - C$

By solving the previous differential equation associated to our biomass recurrence relation, we obtained different formulas for the biomass of the n-th generation.

Making a discussion on the value of C and on the relationship between Cr and $\frac{M}{4}$, we obtained four different formulas for X_n, all depending of the generation number n, M, r, C and X_0.

From computing and graphing several particular cases (like the ones in the graphs from the previous sections), we saw that the majority of them lead to extinction.

However, if $C = 0$, whenever we get a X_k very close to M (or 0), the variation between generations is almost null, meaning that the sequence converges to M (or 0) and from a certain n, the sequence will be constant (as we illustrated in one of the previous sections).

Next goals of the research:

- Introducing the variable V_n

We will study the new recurrence relation by dividing the new problem in cases concerning the value of V_n, and its relationship to V_n, V_n, and V_n. If V_n is a constant, then the method of use will be almost identical to what we have used, and the formulas will also look similar to the previous ones.

- Understanding unusual cases

While our majority of examples led to extinction, we also discovered sequences with a peculiar behaviour, such as the one below. We want to understand why this growth is uncontrollable.