Untwisting braids

2018-2019

Students' names and grades: Dragos Crisan, Alisa Maier,12th grade

Institution: Colegiul National "Emil Racovita", Cluj-Napoca

Teacher: Ariana Vacaretu

Researcher: Lorand Parajdi, "Babes-Bolyai" University

1 Topic presentation

A braid consists of n vertically arranged yarns that intermingle in various crossings and are tied up and down. If we can remove all the crossings of a braid without cutting wire or moving their ends, we say that the braid is trivial. Opposite, we see two braids with three yarns, on the left a trivial one and on the right a non-trivial one:
Borromean tresses Are there non-trivial braids such that if we remove any of its strands (for example if we cut it) then the braid becomes trivial ?
Anti-braids Is it possible to stick under a braid another braid such that the braid obtained is trivial?

2 Results

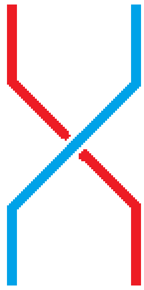
We found an example of a Borromean braid and we proved that it is non-trivial using an invariant, namely tricolorability.
Also, we defined a group explaining the structure of a braid and we tried to study its properties.

3 Text of the article

3.1 Notations

Label the yarns $1, 2, 3, 4, \dots, n$.

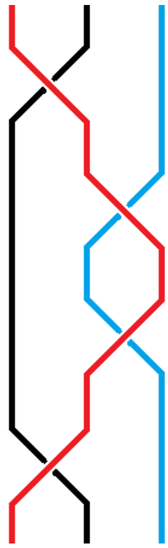
If the yarn i crosses the yarn j such that i is under j, we denote this by (i, j), so the entire braid can be denoted by a sequence of ordered pairs.



(i, j)

i goes under j

Here is an example of a trivial braid:



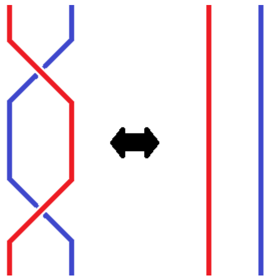
given by the following sequence:

$$(2, 1)$$
$$(3, 1)$$
$$(3, 1)$$
$$(2, 1)$$

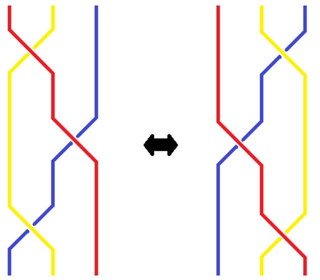
This notation is useful in showing that, sticking a braid S' under a given braid S, the resulting braid won't be trivial. If there exists a sequence of operations which, applied on $S \cup S'$, gives a trivial braid, select from this operations those which are performed only with yarns in S. These give a sequence of operations which untie S.

3.2 Axioms

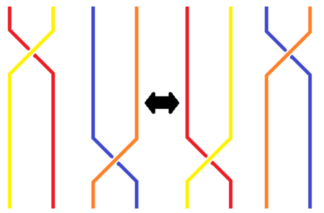
There are 3 types of operations (axioms) one can perform on a braid:
Axiom I



Axiom II

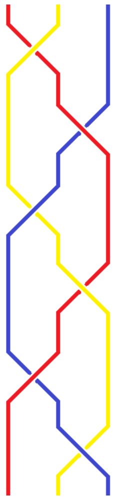


Axiom III



3.3 Borromean braids

The following braid is non-trivial, but removing any of its yarns yields a trivial one:



(1, 2)

(3, 1)

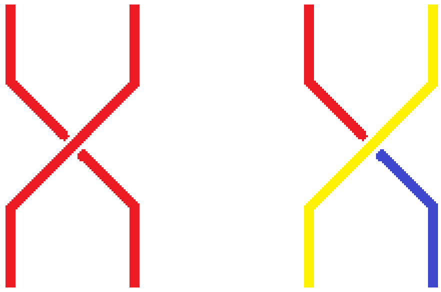
(2, 3)

(1, 2)

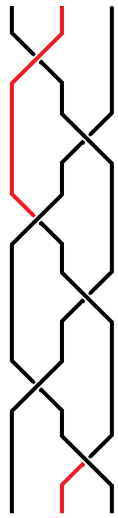
(3, 1)

(2, 3)

It is easy to check that removing any yarn leaves a trivial braid. It is harder to prove that the initial braid is non-trivial. For this, we will define an invariant, some property that doesn't change when we apply our axioms. As in [1], we call a braid **tricolorable** if it can be colored with exactly three colors, subject to the following rule: at every crossing, the three parts of yarns have either the same color or three different colors. (note that we require that the beginning and the end of every yarn to have the same color).

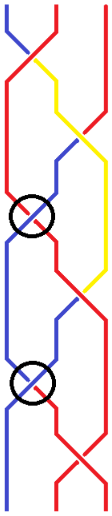


By checking, we can see that the property of a braid being tricolorable is invariant under all the three axioms. That is, if a braid is tricolorable, and we perform any of the three operations, the resulting braid will also be tricolorable. As the trivial braid is tricolorable, it is enough to prove that our braid is not. For this, we just try to color it:



Looking at the top-most intersection, there are two cases: the three parts have the same color or different colors. We continue to color the different parts of yarns. The first case leads to a contradiction (there is only one color used):

The second case leads to another contradiction:

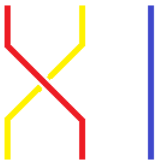


3.4 The group

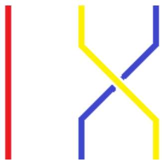
A further topic for research could be the group associated to a braid. Given a braid with three yarns, we define its group to be

$$G = < x, y >$$

where x and y have infinite order.



x



y

The first axiom is equivalent to x and y being invertible. The second axiom gives a relationship between the generators:

$$xyx = yxy$$

This defines:

$$G = \{< x, y >: xyx = yxy\}$$

In order to prove that our braid is, indeed, non-trivial, we should prove that $\left(x^{-1}y\right)^3 \neq e$, where e is the identity.
Also, note that this group structure can be extended to n yarns, for any n, the resulting group $G = < x_1, x_2, \ldots x_n >$ having the properties $x_i x_j x_i = x_j x_i x_j$, for i and j adjacent (Axiom II) and $x_i x_j = x_j x_i$, for i and j non-adjacent.

4 Conclusions

Although we managed to find an example of a Borromean braid and prove that it is, indeed, Borromean, we haven't yet managed to find examples with more yarns. It would be interesting to be able to find an algorithm to determine whether a given braid it trivial or not, maybe using the group structure.

References

[1] Kayla Jacobs: Tricolorability of knots,
https://dspace.mit.edu/bitstream/handle/1721.1/100853/18-304-spring-2006/contents/projects/jacobs$_k nots.pdf$