DON'T CROSS THE STREAMS

Radu ILIEȘ, Andrei MAN
Colegiul Național "Emil Racoviță", Romania

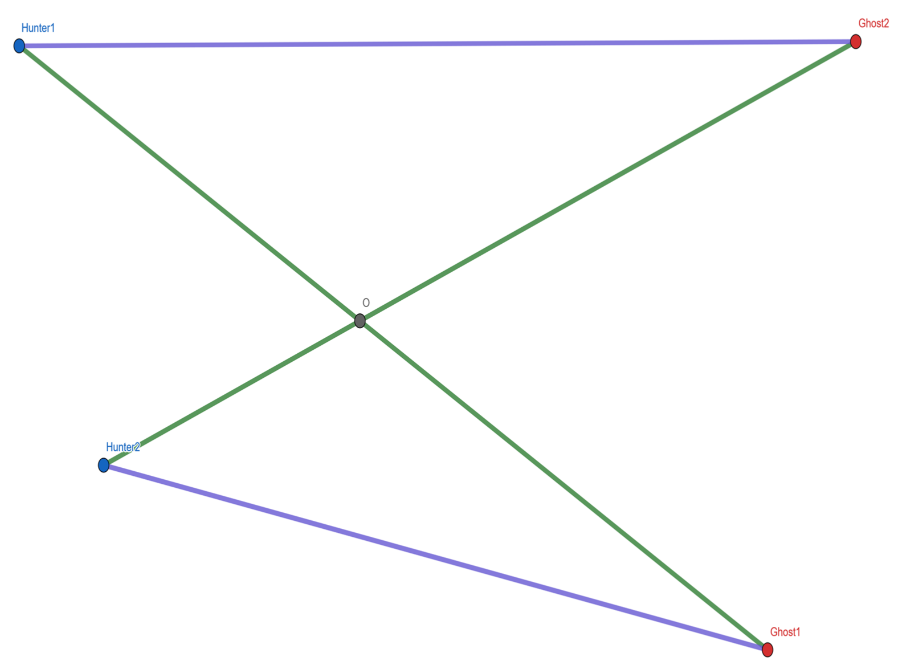
Our research topic

The ghostbusters use rays for neutralizing the ghosts. In the image below, it is essential that each two rays do not cross.

Is it possible for the ghostbusters to distribute in such a way that each one hunts a ghost and any two rays don't cross each other?
Can we find a solution using the computer?
Can we always find such a configuration that each hunter shoots his own ghost and the rays do not cross?

Starting with a simple case

In this situation, we have 2 hunters and 2 ghosts.
It is clear that we can find two lines that do not cross.



Solution

Let $A = \{H_1, H_2, \ldots, H_n\}$ denote the hunters and $B = \{G_1, G_2, \ldots, G_n\}$ be the ghosts.

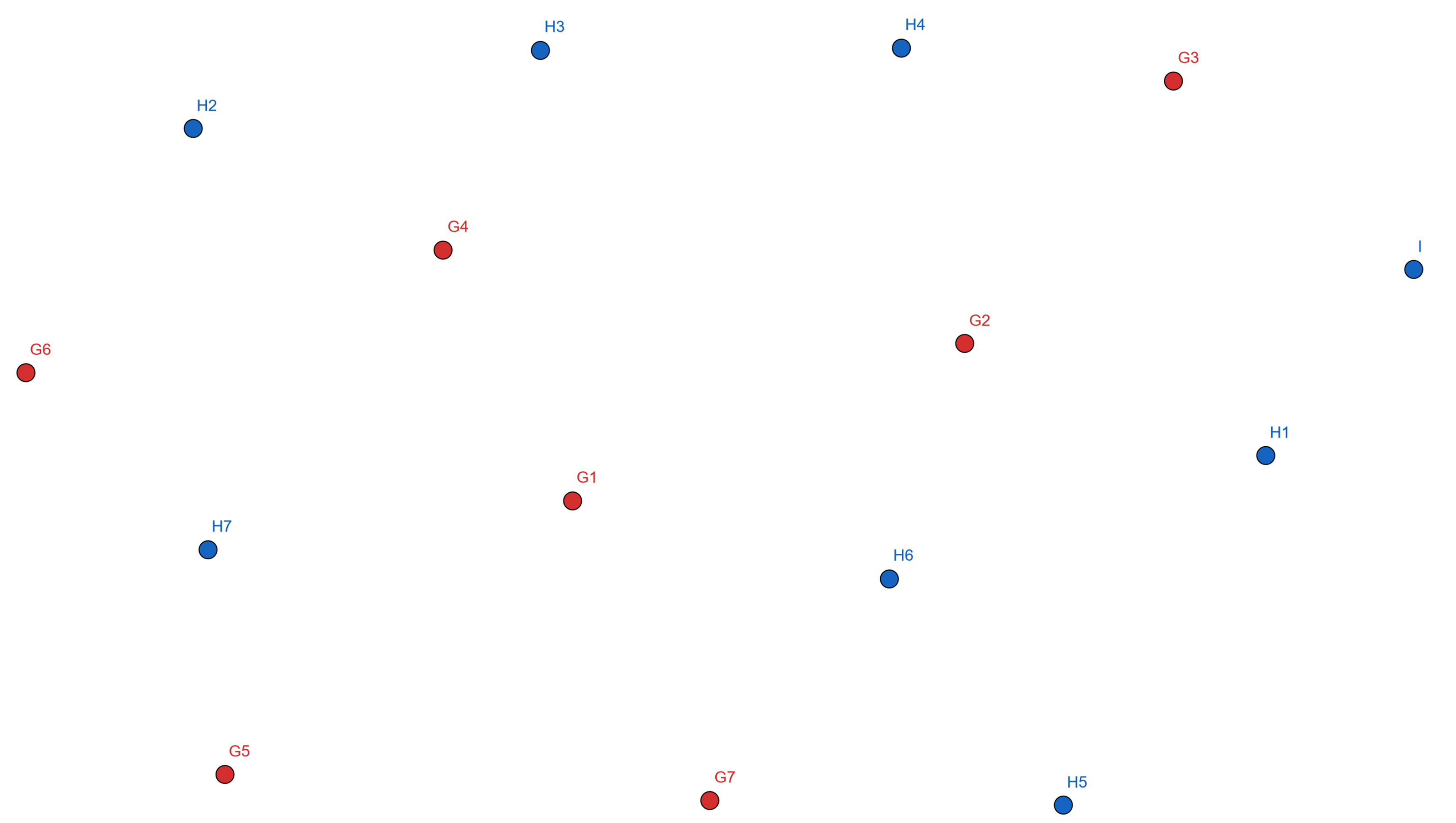
Axioms:
1. For each hunter, there exists a unique ghost.
2. For each ghost, there exists a unique hunter.

Lemma:
We define the one-to-one correspondence $\sigma: \{1,2,\ldots,n\} \longrightarrow \{1,2,\ldots,n\}$.

Let $S(\sigma) = \sum_{i=1}^{n} H_i G_{\sigma(i)}$ the sum of all the distances between the ghosts and the hunters.

There always exists a function σ for which the sum S is minimum. Let σ_0 be this function.

When σ_0 matches the points, the lines do not cross.



Conclusions

Therefore, if the points are not on the same line, we can find a matching such that none two such lines cross.

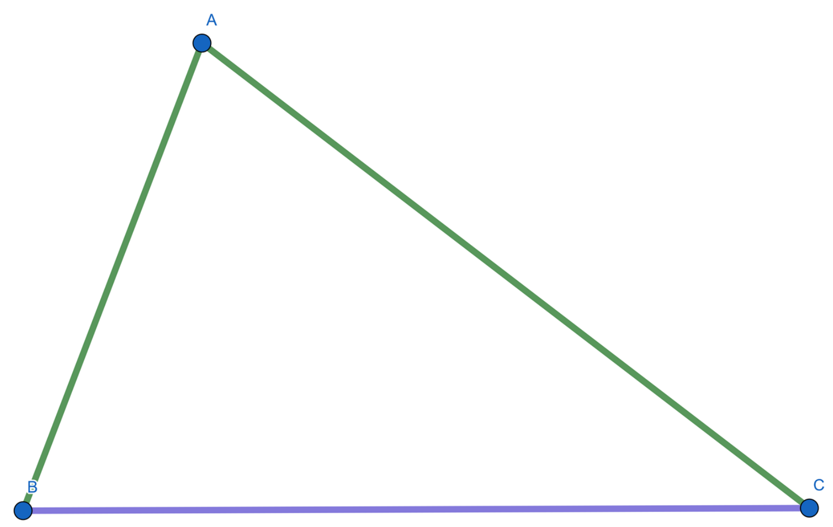
We can find this situation by computing all these sums and finding the smallest one, which obviously exists.

A useful inequality

This is also known as the triangle inequality:

If A, B, C are three points in the same plane, then

$$AB + AC \geq BC$$



How to reach this solution

When trying to solve Math problems, it is always useful to make a plan.

In this problem, the key was to find the minimum value of a sum and see a particular situation in that case.