LICEO SCIENTIFICO STATALE
"B. ROSETTI"
V.le De Gasperi, 141 – 63074 San Benedetto del Tronto (AP)

Erasmus+

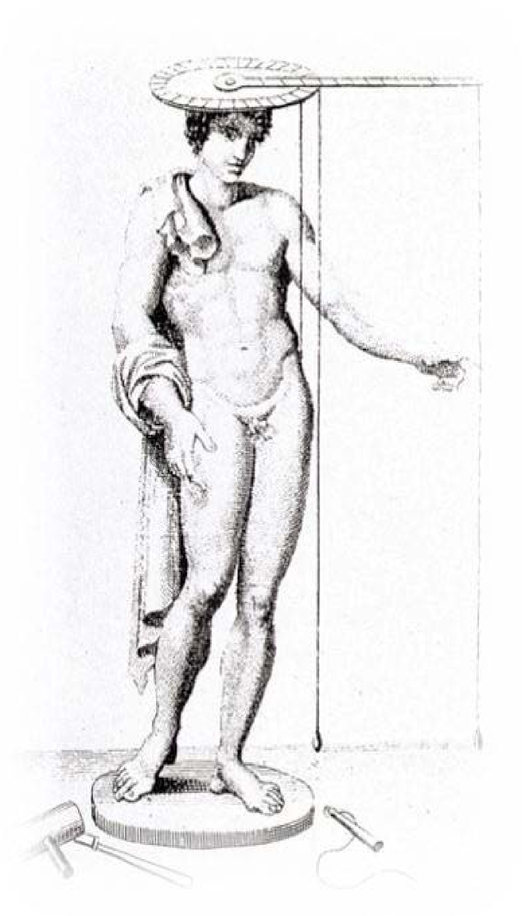
Math around us

Math in Art

San Benedetto del Tronto 25-29 January 2016

Preliminary Remarks

The proposed path has the purpose of analysing the relationships between maths and art. G. Harold Hardy, a great British mathematician, stated that one of the main features of mathematics was beauty and that "*The mathematician's patterns, like the painter's or the poet's, must be beautiful*"[1]. Mathematics and art can be considered an indissoluble couple, in fact, there are numerous binds, facets, convergences and divergences, that can prove this connection. Moreover, there is a strong relationship between figurative art and mathematics as both of them are human creations based on fantasy and on a rigorous language. We would like to study exactly this strictness, a meticulousness that has contributed to create some of the most beautiful works of art and has influenced and moved the excellent minds of the greatest thinkers. The examined themes are: the proportions, the golden section and the perspective, with a particular attention to the theoretical treatises that have supported the diffusion of knowledge and have been essential references for artists and mathematicians. In the analysis of the art history, from the origins to the present day, it is always possible to find that tie, thus it has been decided to focus the attention on fifteenth and sixteenth centuries, with references to classical art, that has always been a source of inspiration during the Renaissance.

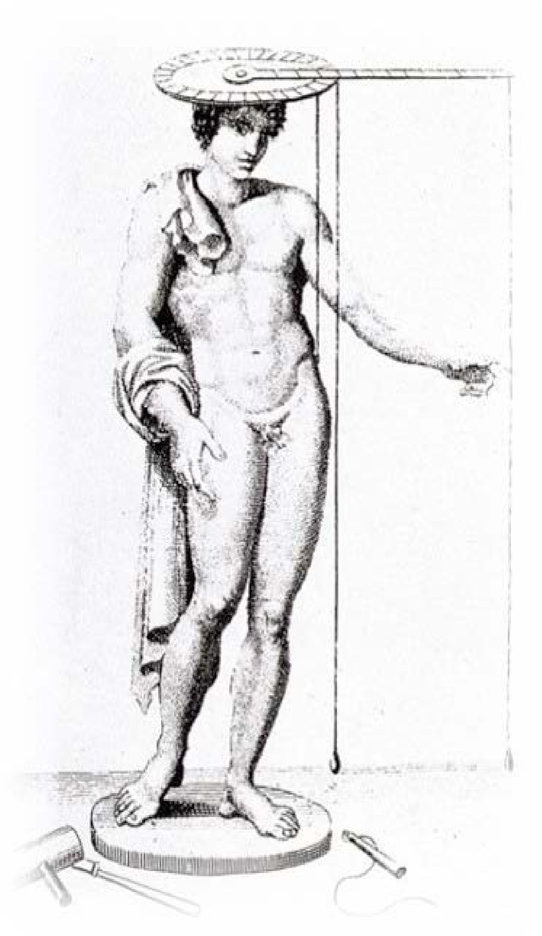


Leon Battista Alberti, measurement of a statue with *finitorium*

1. G. HaroldHardy, http://izquotes.com/quote/234920

Renaissance takes its origins in the literary Humanism of XIV century that promoted the "rebirth" of culture through the **rediscovery of the classical world** (which entails the knowledge and the study of Greek literature and philosophy, abandoned during the Middle Ages). The Humanism searches in the Latin and Greek classical cultures the foundations for the recovery of **Man's values** interpreted in their completeness of spirit, intellect and matter: a new man, able to understand natural laws and to take part in history, a man capable of an active practice of his intellectual virtues and of putting them at the service of the community. For this reason, the formation of the individual must derive from the balance among the different fields of knowledge (literary, philosophical and scientific). The new vision of the world and of man is stated by Pico della Mirandola (1463-1494), an Italian humanist and philosopher, living and working in Florence at Lorenzo il Magnifico's court. In his *Oratio de hominis dignitate* (Oration on the dignity of man), considered the Manifesto of the Renaissance, the philosopher deals with the fundamental theme of the freedom of man that is exalted as a creature able to know and rule reality:

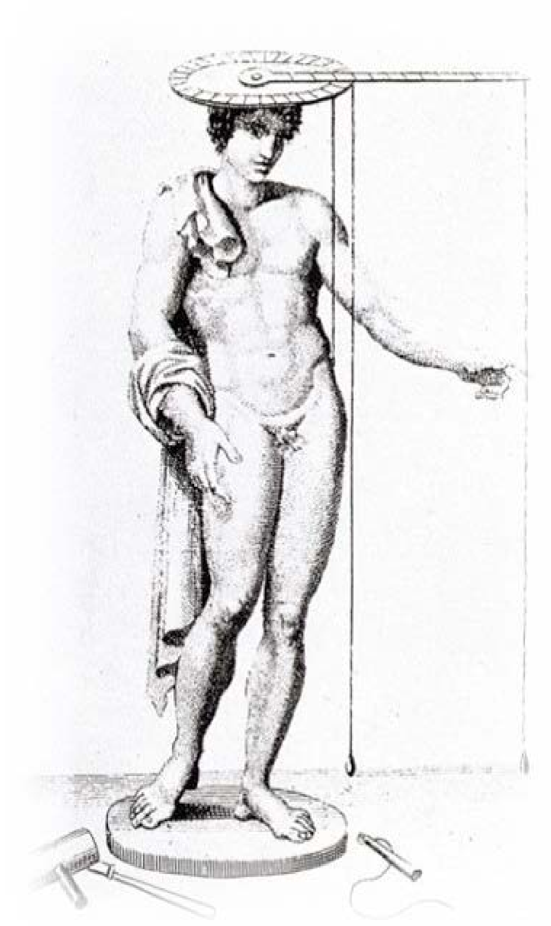
"*But when this work was done (* the world creation*), the Divine Artificer still longed for some creature which might comprehend the meaning of so vast an achievement, which might be moved with love at its beauty and smitten with awe at its grandeur. When, consequently, all else had been completed […] in the very last place, He bethought Himself of bringing forth man[…] We have made you a creature neither of heaven nor of earth, neither mortal nor immortal, in order that you may, as the free and proud shaper of your own being, fashion yourself in the form you may prefer. It will be in your power to descend to the lower, brutish forms of life; you will be able, through your own decision, to rise again to the superior orders whose life is divine.*"[2]



2.Giovanni Pico della Mirandola, On the Dignity of Man , http://www.andallthat.co.uk/uploads/2/3/8/9/2389220/pico - oration on the dignity of man.pdf

In the figurative art, in fact, **the correct proportions of human body** are studied and considered measurement reference for everything, as mirror of perfection of all creation. The artists of the Renaissance needed to have remarkable technical skills and notions related to those branches of knowledge connected to art like mathematics, geometry and optics , thus *Mechanical Arts* acquired the same dignity as *Liberal Arts.*

In the analysis of the features and the themes of the Renaissance it is clear the necessity of mathematical knowledge. The main themes are: *the rediscovery of classical art (Greek and Roman,* source of knowledge and normative reference for the new artistic language in which the ideals of order and rationality are expressed), *the central role of man*, *the establishment of a new concept of beauty (*based upon the application of proportional harmonic rules) and the *elaboration of the linear centric perspective (*used as representation of the space but also meant as a symbolic form). A new ideal of beauty is defined, conceived as **harmonious and well-proportioned connection of forms** which results in the clarity of geometry and of mathematics correlations, in the order given by the correspondence of harmony between the parts.

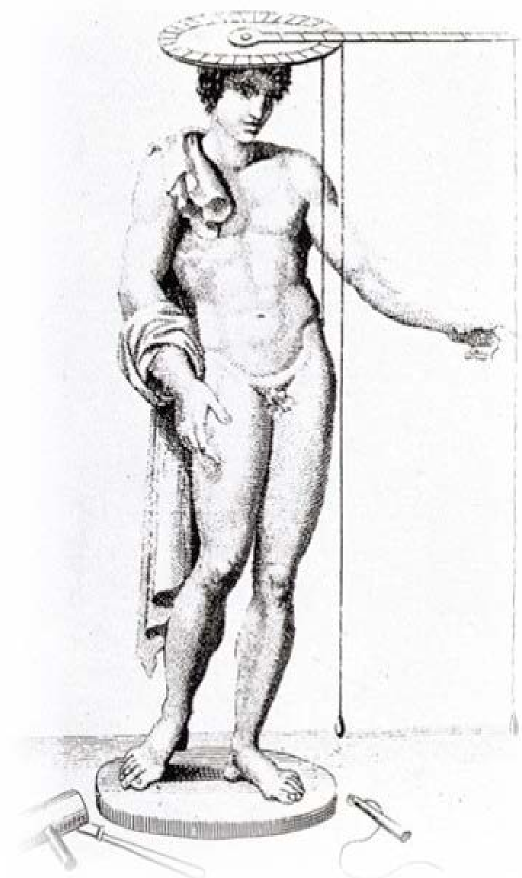


“The senses are delighted by things with the right proportions”

Saint Thomas Aquinas (1225-1274)

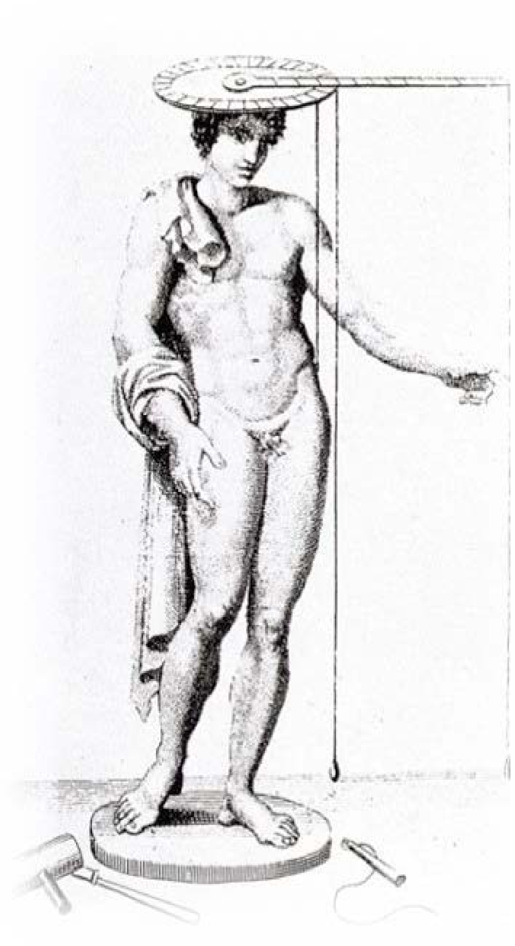
“Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel.”[3]

Johannes Kepler



3. Johannes Kepler, http://www.goodreads.com/quotes/tag/pythagorean-theorem.

The new ideal of beauty that is stated during the Renaissance is based on the harmonious and proportioned accord of the forms. The term proportion derives from the Latin *pro portione* that means “according to the section” and refers to the correspondence of measures between two or more parts strictly related. Thus, these correspondences of measures are mathematical relationships. The proportions are applied to architecture that bases its compositional order and structural calculations on the exactness of the number.



LINEAR AND STATIC RELATIONSHIPS

Greeks appreciated geometry even for its theoretical interest, in fact, some philosophers like Pythagoras (570 -490 BC) and Plato (425-327 BC), considered geometry an important intellectual instrument and for its purity and abstract nature they believed it akin to metaphysics and religion. Around 300, BC Euclid collected in a systematic form the main geometrical discoveries of his forerunners in the essay "*Elements*" (in 13 books) that has influenced the entire Western scientific thought.

From the study of numerical laws which regulated the musical harmony, the Pythagorean school discovered some morphological principles of general nature (general principles), which soon became the compositional rules of every type of art, in particular that which dealt with the construction of sacred structures. Greeks noticed that if we make vibrate two taut strings, one of which is twice the length of the other, the sound of the shorter will be one octave higher than that of the longer one. This numerical ratio, in Greek *diapason*, is written 1: 2 (one to two). The audible harmony, generated from a set of notes well combined, is transformed into visible harmony through well-designed forms in an architectural building.

The numerical ratios most commonly used are the *unìsono* (1: 1), the *diapason* (1: 2), the *diapènte* (2: 3), the *diatèssaron* (3: 4). As a result, designing or creating a building facade whose height is twice its width will create a harmonious construction that makes the musical harmony of the *diapason visible*.

Music was studied in relationship with mathematics, astronomy and geometry.

The Canon and the Harmonic Proportions

In order to research and create beauty it was necessary to find the right numerical ratios among the parts of an object or of human body. The revival of the classical formal models corresponds to the recovery of the *canon*, that is a formula, a set of rules which the artist depends on. The *canon* is a system of proportional relations among the whole structure and its single parts and between the structure and the space. The architectural *canon* had a mathematical nature and was applied both to buildings in their whole and to their parts as columns, capitals, cornices and tympanums. The *module* was at the base of everything, in other words it was an element established as reference measure for the proration of the building (generally in Greek architecture the module was the diameter of the column at the base : the lower scape, or sometimes the space dividing two columns : intercolumniation). The proportions derived from the repetition of the form (multiple) or its division (submultiple) and were expressed as ratios of integers or fractional numbers.

In mahematics an integer **a** is **multiple** of another integer **b** if there is a third integer **c** such that moltiplied by **b** results in **a**. Thus **a** is multiple of **b** if and only if a **c** exists such that **a = c** x **b**

The product of two integers is called **integer multiple**

Submultiple of a quantity is each quantity contained an integer of times in the given quantity : 3 is a submultiple of 9 (See exercices)

Filippo Brunelleschi (1377-1446) skilled Florentine architect, enthusiast of mathematics, painting and sculpture, great connoisseur of the classical art shows interest in such proportions, in fact, in the construction of the *Spedale degli Innocenti* (The Hospital of the Innocents) he applies a clear modular principle based on a simple geometrical logic. Since the second decade of the fifteenth century, his activity focuses on the architectural design and on the theoretical reflections in the field: we owe to him the elaboration of a new project system, based on geometry and on the modularity of the structures, in addiction to the "discovery" of the central linear perspective and the recovery of the principles of classical architecture.

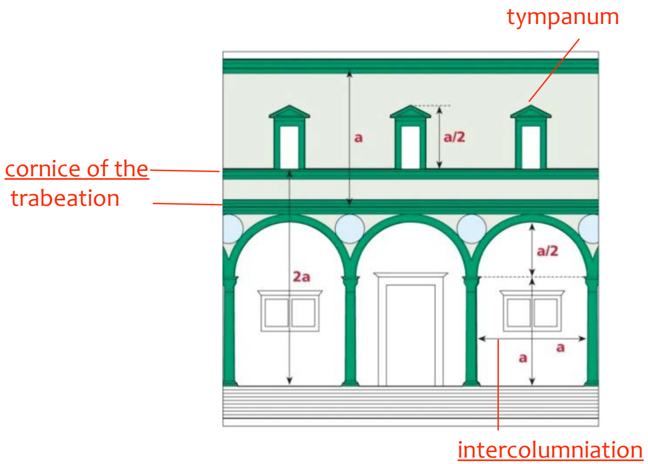
The ideal of beauty expressed by Brunelleschi is strictly connected with the aesthetical principle of the classical measure together with the clarity of composition and with the order among the parts granted by the adoption of the *module*, which determines the dimensions of the totality for multiples and submultiples. The clarity of Brunelleschi's architecture also depends on the use of linear geometrical elements like columns, trabeations, archivolts, that are highlighted by the use of "pietra serena" (a bluish sandstone used in Tuscany) . The architectural orders, the mouldings, the tympanums, the columns and the pilasters strips are composed by creating rhythms based on the basic geometrical forms of the circle and of the square.

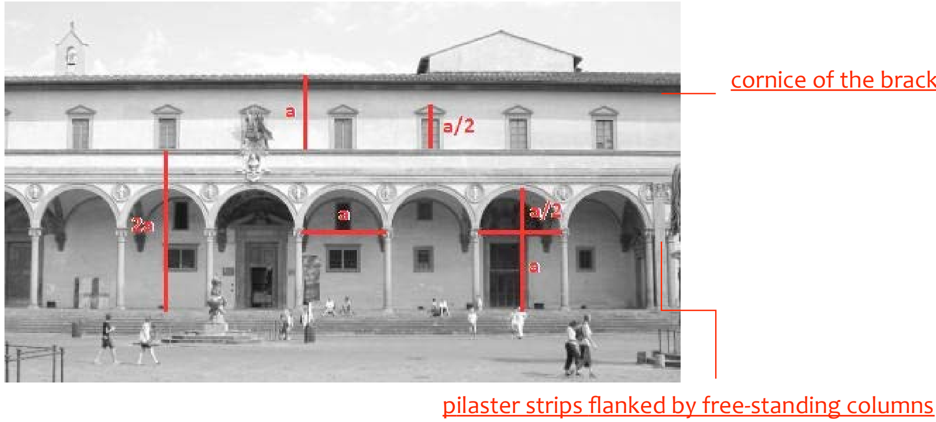
Filippo Brunelleschi, Lo *Spedale degli Innocenti*, 1419 – 1444, Florence

To this work, *Lo Spedale degli Innocenti* (The hospital of the Innocents, so named because in the Hospital children abandoned by young unmarried mothers were hosted) begun in 1419, Brunelleschi worked until 1423, then other artists completed it. The building stands on a staircase consisting of nine steps and nine are the arches of the porch located in the lower portion of the building.

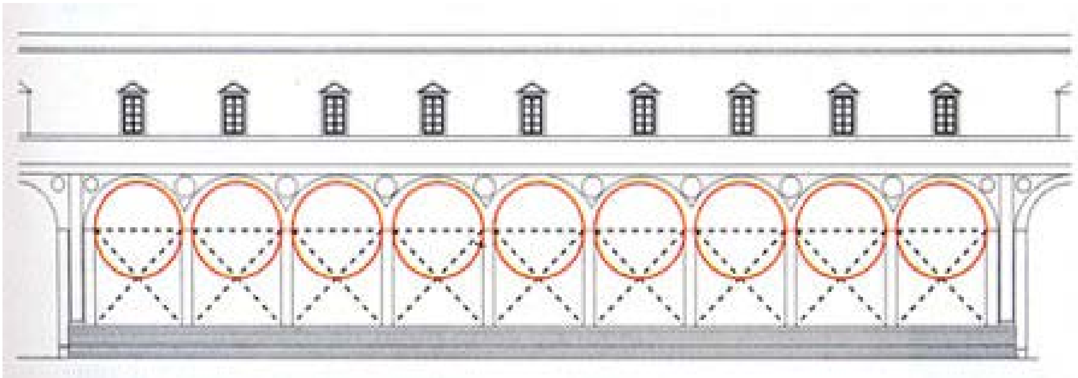
Nine are the bays covered by ribbed vaults and nine again are the windows of classical form surmounted by a tympanum and leant against the cornice of the high trabeation that touches the round arches in one point. At the ends, two pilasters are flanked by free-standing columns. The intercolumniation is equal to the height of the columns and the depth of the porch, consequently the span appears to be cubical in shape.

The space of the open gallery is modular. (See exercises)

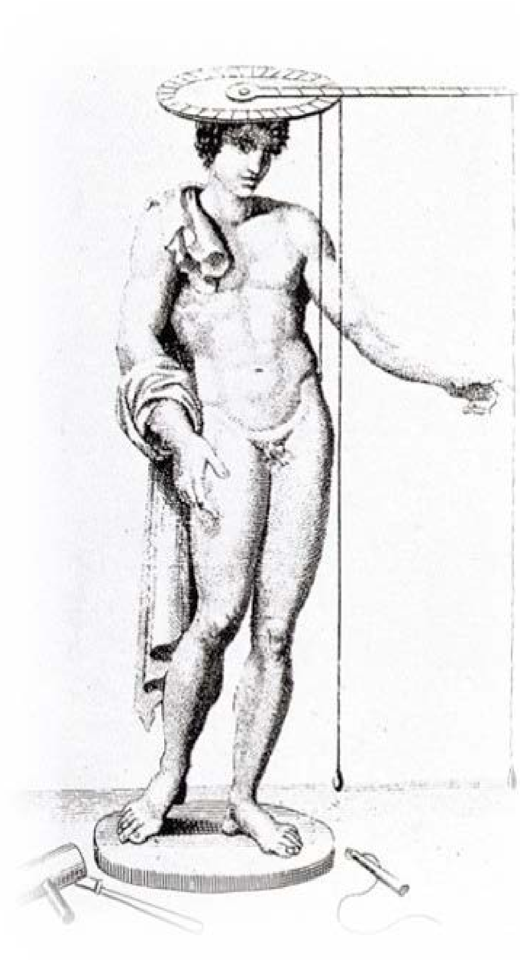




The space of the open gallery is modular. The distance between the floor and the cornice of the trabeation is twice the height of the column. At this height corresponds the distance between the extrados of the architrave and the intrados of the cornice of the bracket. Half the height of the columns is the overall size of the windows, from the window sill to the vertex of the tympanum.



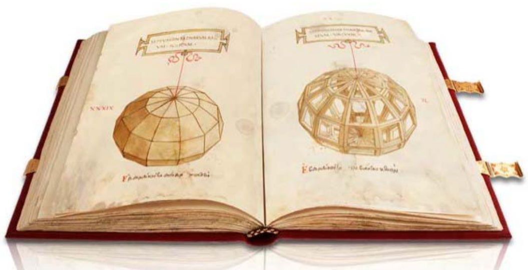
In addiction to **linear** and **static relationships** derived from integers and commensurable numbers, Greeks also knew **dynamic correlations** which produced irrational values. The latter, as irrationals numbers were not known, were obtained through geometrical procedures. During the Renaissance this proportional rule was defined "divine proportion" which corresponds to the modern definition of golden section.



THE GOLDEN SECTION

Nowadays more than ever the world we live in is passionate about numbers, in particular there is one among these worthy of attention. It is an irrational number, particularly interesting: 1,618033988. This figure has fascinated throughout history brilliant minds. For centuries this number has been called by seductive names as golden number, transcendental proportion, divine number, divine proportion, etcetera. The golden number, which is represented with the Greek letter ϕ (phi), belongs to a region made up of connections and unbelievable and unexpected numerical properties. One of the wonders of the golden proportion is its unlimited capability of creating figures of great beauty and bewildering properties as the right-angled polygons (whose sides respect each other the divine proportion) or the regular polygons (like the pentagon and the various figures that originate from it), or else the spiral. The divine number is mysteriously linked to concepts of beauty and perfection, in fact, for some unknown reasons, all the geometrical figures that respect this proportion are particularly pleasing to the eye. Probably the golden number was discovered by the Greeks (but perhaps used even before), the symbol (Phi) ϕ, with which today we identify it, was given just at the beginning of the XX century when the American mathematician Mark Barr suggested that the number should be linked to Phidias, taking the initial of his name.

Jacopo de' Barbari, *Portrait of Luca Pacioli and of Duke Guidobaldo da Montefeltro*, 1494, tempera and oil on panel, 99 x 120 cm. Museo di Capodimonte, Naples

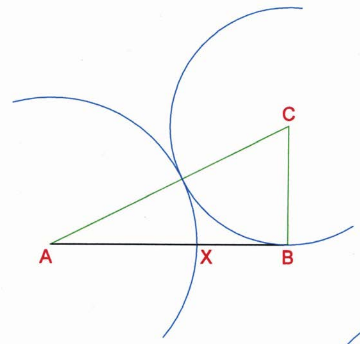


Illustrated text: *The Divine Proportion*

Euclid's definition of the golden ratio is the following: "*Straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less".* The famous golden number shows exactly this mean and extreme proportion. The ancient architects had to achieve the Symmetry ("harmony of measures") through the repetition of certain privileged proportional ratios which would have produced and characterized the Eurhythmy ("harmony") effect among the lengths, the surfaces and the volumes of the construction, in its entirety as well as in its single parts. The compositional technique was that of regulatory plans, of the refined geometrical constructions that started from an initial form, the square, to identify, through simple projections and overturnings, all the principal lines of the construction, in the plans and in the elevations.

In 1509 Luca Pacioli, an Italian religious , mathematician and economist, devoted an entire treatise to the golden number, *De Divina proporzione (* The Divine proportion). It is an immense work on Euclidean geometry that deals with the golden section, the architecture of the regular polyhedrons and of those derived from them.

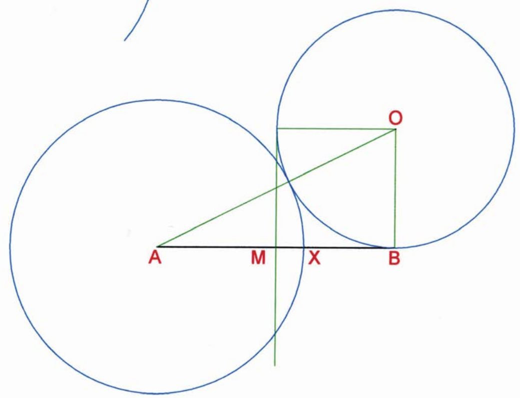
In geometry the term *proportion* means matching measurement between two elements in relation to each other; the proportion of a geometric figure, then, is defined as the relationship of the ratio among the parties and it is identified with the mathematical formula A : B. It usually has two meanings, one quantitative and the other aesthetic, because despite being mathematically defined, it also has the ability, if applied to objects that strike the senses, to make them beautiful and pleasantly harmonious.



Calculation of the golden section given a segment AB (Ex. 4,9,11, Table 2)

The golden ratio is the link between a segment and its part which is proportional mean between the segment and the remaining part. In this section the length of the two parts A and B is mean and extreme ratio of the entire segment A + B; it is the only section in which A: B = B: (A + B). Given the value 1 to the segment AB, the section AX will be an irrational number that has the approximate value: 0,618033988

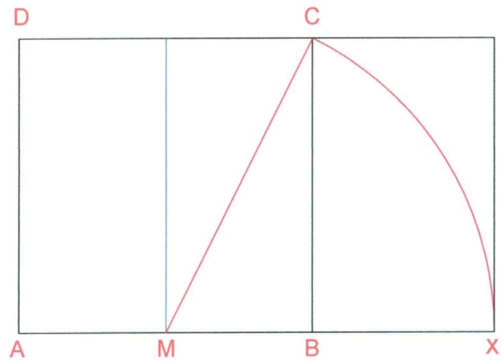
AB : AX = AX : XB



Construction of a golden rectangle (Ex.2, Table 5- Ex.6, Table 3)

The **rectangle** that can be obtained from the transformation of the square (regular geometric figure which is characterized by a **static nature** as it retains all the symmetrical properties and it is based on the modular repetition of its elements: the side and the diagonal) is a geometric figure which is marked by **dynamism** as it develops the two directions of the axes according to the **proportions** and not to the equality. A rectangle whose sides correspond to the values of the golden section is defined “golden rectangle”. Adding or subtracting to the golden rectangle a square whose side has the same length of the longer side of the rectangle, an infinite sequence of rectangles with 1,618 increase is generated.

If you draw a square of height 1 divided into two rectangles by the vertical median, the diagonal of one of the rectangles will have ratio 5 and overturned on the extension of the side, it will define the golden rectangle

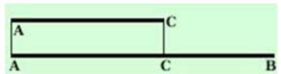


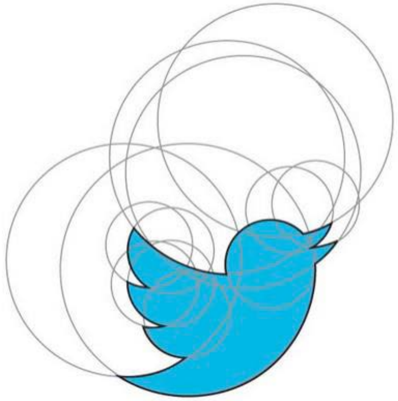
Definition

The **golden section** of a segment **AB** is the part of the segment that is the mean ratio between the segment and the remaining part.

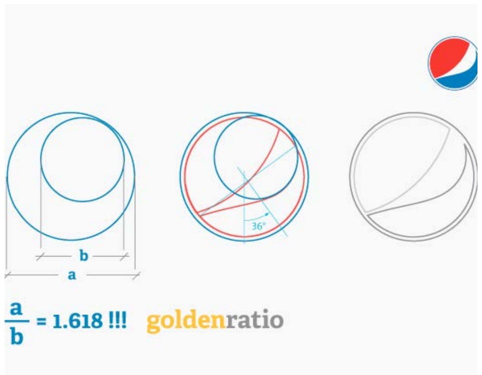
AB : AC = AC : CB

A point C must divide the segment in such a way that the segment AC is in the mean ratio between the segment and its remaining part CB.

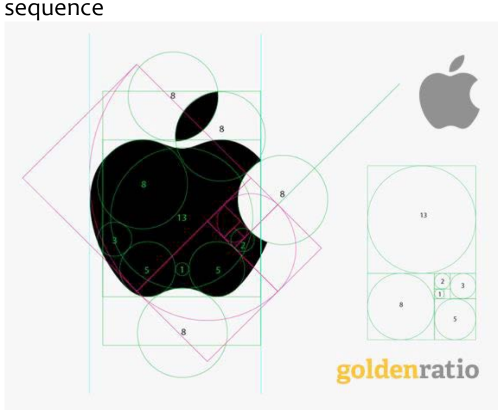




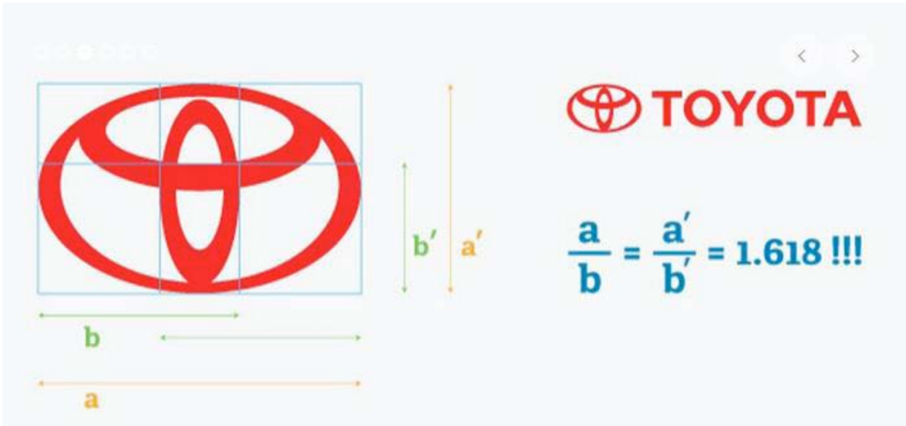
The bird of Twitter was created by **overlapping circles** with diameters equal to the numbers of the Fibonacci sequence

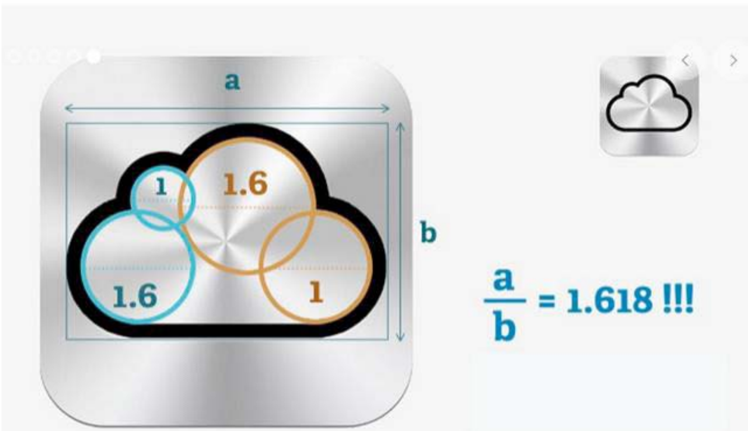


The Pepsi Cola logo is composed also by some harmonious and “golden" geometric shapes



The Apple logo is made up entirely of circles whose rays give the numbers of the Fibonacci sequence





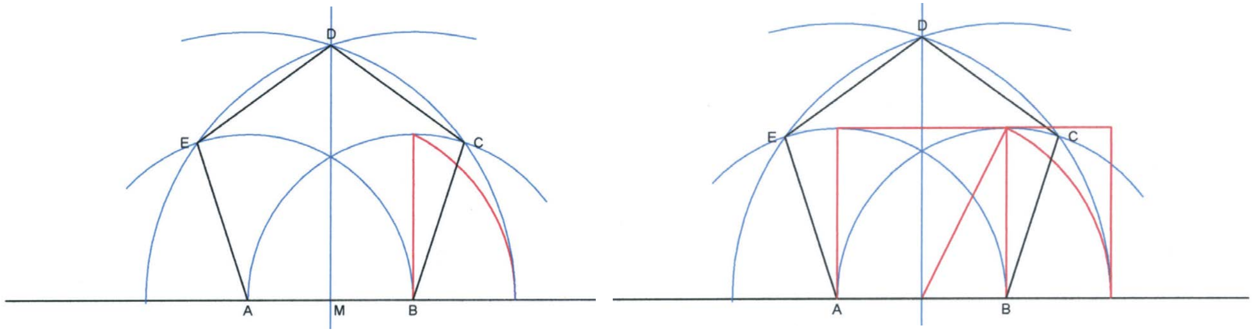
Icloud symbol is contained in a golden rectangle and the diametres of the circles that create it are in golden ratio

THE GOLDEN NUMBER AND THE PENTAGON

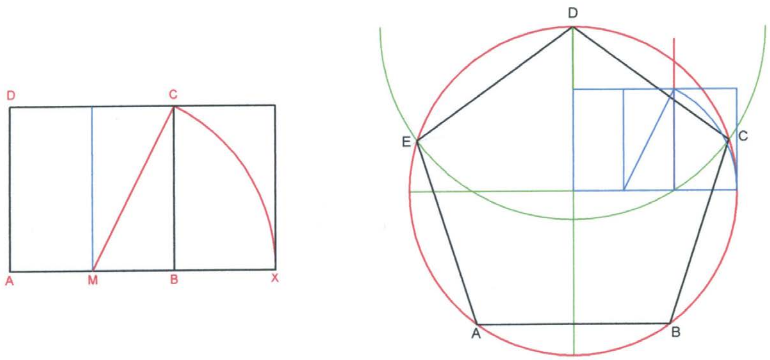
The golden ratio in Euclid's exposition can be found in the construction of the pentagon and of some regular solids. Among the regular geometric figures the richest of relationships with the natural world is the pentagon. In fact in its internal structure it contains the same principles of the golden progression. Analysing the internal structure of the pentagon, a great number of figures can be discovered, all of them ascribable to the golden ratios, because the side and the diagonal of this figure are two segments in golden ratio. The pentagon is constructible with straightedge and compass making use of ϕ.

CONSTRUCTION SYSTEMS

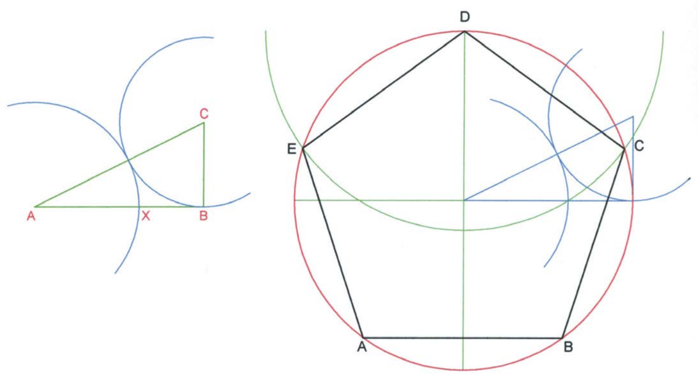
Construction of the Pentagon given the side AB, using the golden rectangle (Table 3, Ex.7)



Construction of the Pentagon given the side AB using the division of the segment in mean and extreme ratio.

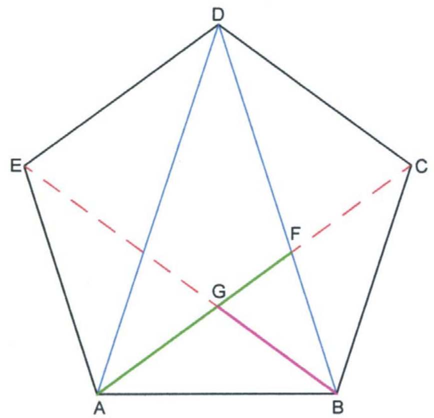


Construction of the pentagon inscribed in a circle, using the division of the segment in mean and extreme ratio.



The golden number and the regular pentagon

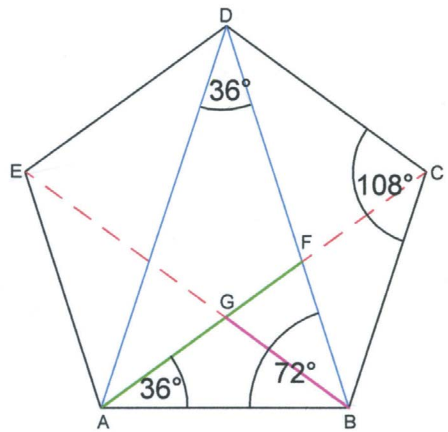
Looking at a regular pentagon on which the diagonals were drawn and examining the isosceles triangle ABD, we may notice that the diagonals DA = DB are in the golden ratio with the base AB. Tracing the bisector of the angle A we obtain the triangle ABF. This has the same angles as the original ABD and therefore both are similar. If we continue with this process, bisecting the angle A, we will have the triangle BGF which will be similar to the previous two. Hence we can still say that the relationship between the diagonal and the side of the pentagon is ϕ.



The golden triangle

From the construction of the pentagon it is possible to obtain the golden triangle, that is an isosceles triangle with base angles 72 ° and the other angle 36 °. The base is the golden ratio of the equal sides. The pentagon and its diagonals form two kinds of isosceles triangle. The first has the angles 36 °, 36 ° and 108 °, while the second 36 °, 72 ° and 72 °, in both cases, the ratio of the longest side to the shortest is equal to ϕ.

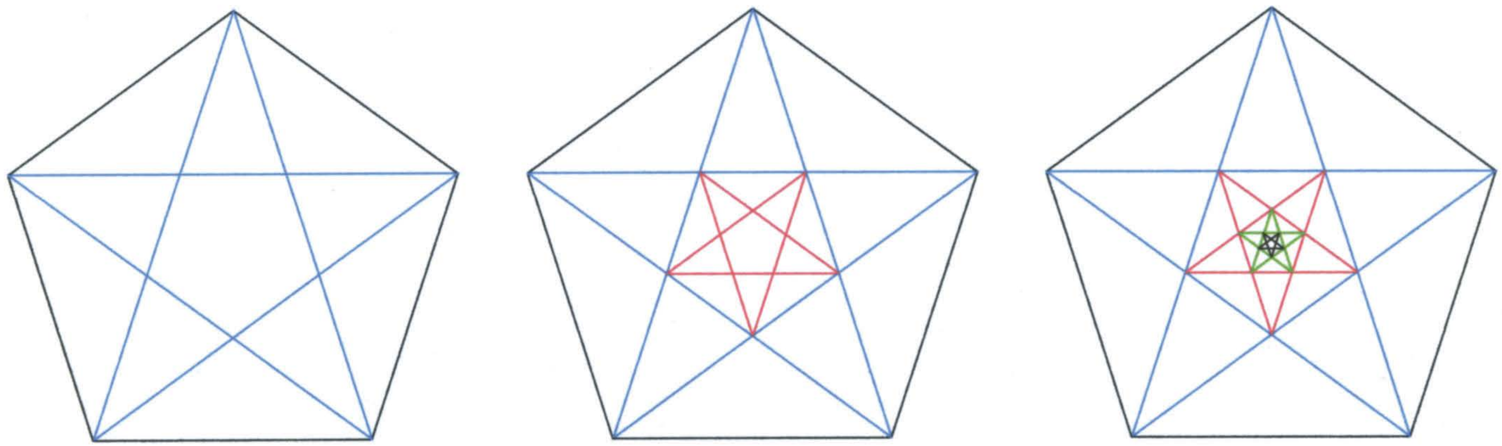
If we take into account the pentagon and the triangles that are created by drawing the diagonals, we can see that there are only three different angles: 36 °, 72 ° and 108 °. Since 72 is twice 36 and 108 is three times 36, all of them result multiples of 36.



With graphic procedures it is possible to construct the Pentagon and the figures linked to it. The pentagram, in fact, is based on the pentagon and on its diagonals; moreover, inside the pentagram, an infinite series of pentagons and pentagrams can be reproduced.

Construction of the pentagram and of the pentagonal star

The representations of stars as pentagonal stars are very ancient, in fact, they have been found both on Mesopotamian tablets and on Egyptian hieroglyphics. The symbol of the star pentagon, also called pentacle, then, became the distinctive sign of the Pythagoreans and was used to identify the members of that philosophical current. According to them the number five was the number of the harmony in health and in beauty as it presumed a balanced combination between the two, the first even number, also called dyad, and the three, the first complete odd number or triad. The pentacle has a long history as symbol and it is also a recurrent figure in our daily world as graphic image to which different meanings are attributed.

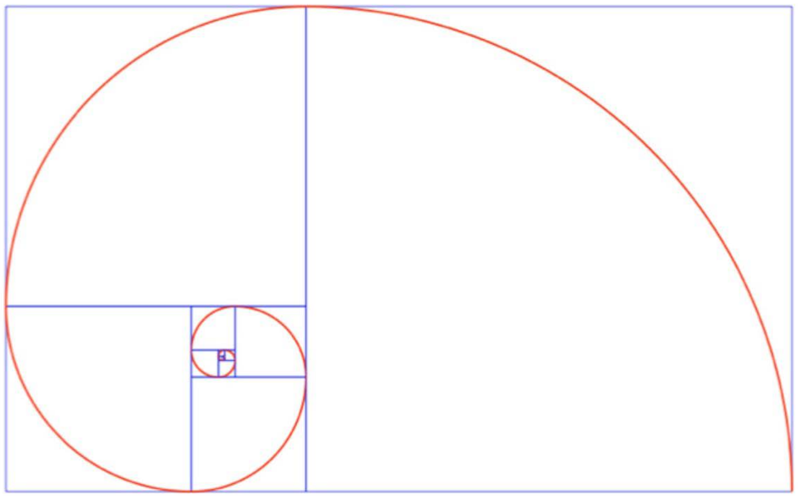


SPIRALS AND THE GOLDEN NUMBER

Some of the most beautiful manifestations of ϕ can be found in spirals, where the golden proportion has a very interesting nature.

The spiral constructed with the golden rectangle

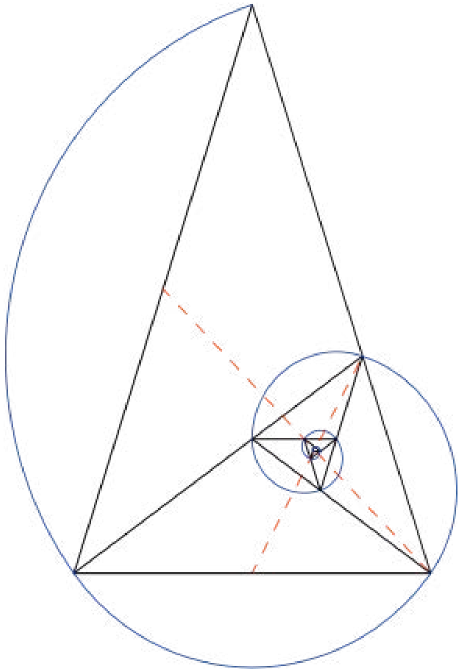
The logarithmic spiral is set to a series of golden rectangles. Starting from the golden rectangle and subtracting it the squares it is possible to obtain a series of golden rectangles. The subtracted squares are used to draw circumference arches, having as radius the side of the square and as centre a vertex of each of them. If we continue indefinitely with the succession of the squares we get the so called logarithmic spiral.



The spiral constructed with the golden triangle (Table 4, Ex.8)

In the case of the golden triangle it is possible to continue to draw bisectors that will form other golden triangles smaller and smaller inside the first triangle. In this way a sequence of golden triangles that will contribute to the construction of the spiral will be obtained.

The infinite reproducibility of these geometric figures gave them a very high symbolic power connected to the cosmos and the divinity.

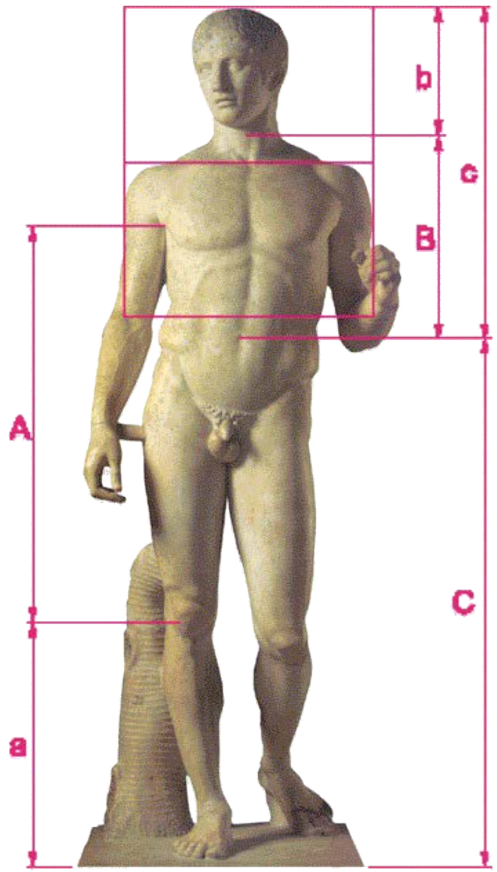


The Golden Perfection in Art

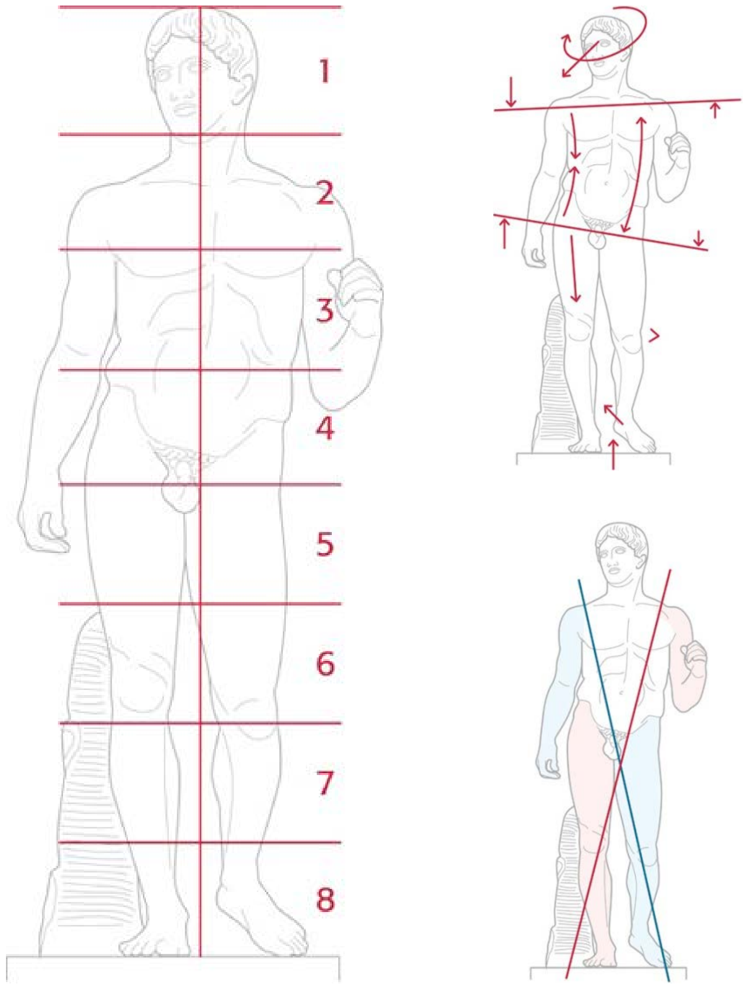
Polykleitos, *Doryphoros*, ca 450 BC.
Roman marble copy of a bronze original, height 212 cm, Archaeological Museum, Naples

The influence of the golden section and its several expressions can be already found in the classical Greece, but its historically registered connection with art started during the Renaissance, with the beginning of a rigorous theorization of the creative act.
The particular passion of the Greeks for the geometrical harmony has shown, at times illegitimately, the presence of the golden ratio in several works of art of the antiquity.
Polykleitos, The Elder (Argos - V century BC) was a great master of the Peloponnese school in the golden age of classical art , in particular he stated the ideal of the human figure. The artist collected his precepts of statue making art in a treatise that he titled *The Canon* (from the greek Kanòn, rule or norm). In the definition of beauty he did not draw inspiration from the severity of gods, but from the human figure and its beauty. Man, central subject of his sculpture, is represented through the combination of ideal forms. The lineaments are clearly based on the observation of actual models but they do not result in portraits as they are referred to models of **ideal proportions**.

In his famous sculpture *Doryphoros* both the *canon* rules and the golden ratio norms find applications.



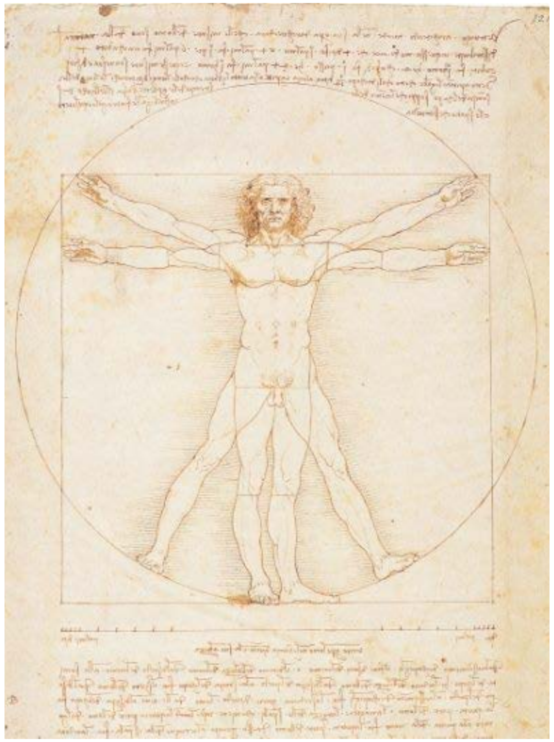
The Golden Perfection in Art



Polykleitus suggests the subdivision of the human figure in eight parts, each one as high as the head dimension.

According to Polykleitus's *canon*, every element of the human body must be represented proportionally to all other elements. The bust must corresponds to three heads and the legs to four heads (in fact 1+3+4=8). His work, considered revolutionary by his contemporaries, introduced the scheme of the X balance (chiasm), that is the inverse correspondence among the parts of the body (in particular between opposite arm and leg).

From this scheme derives the principle of the **ponderation** that indicates the harmonic coordination among the various limbs, in a natural distribution of weights. The artist brings together in a single statue both the sense of movement and that of stasis.



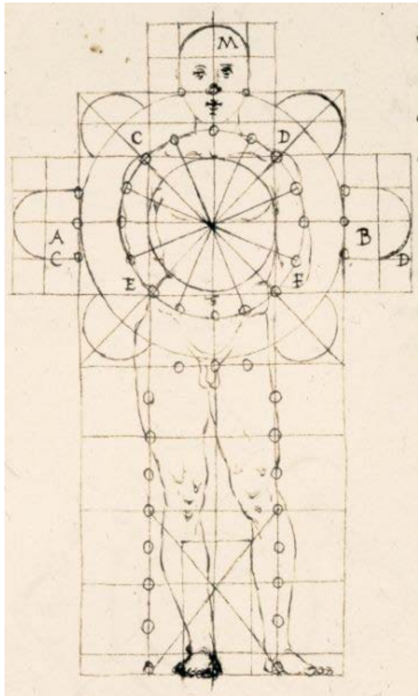
Leonardo Da Vinci, *The Vitruvian Man*, ca 1478 - 1490, pen drawing on paper 34.4 x 24.5 cm, Venice, Galleria dell'Accademia. The drawing shows the ideal proportions of the human body by relating it to the geometry and inscribing it in a square and a circle. The ratio of the side of the square to the radius of the circle is golden.

In order to describe the concept of proportion Vitruvius in his treatise *De Architectura* (On Architecture, I century BC) refers continuously to the example of the human body, and in the III book he writes about the symmetry and the proportions of an architectural building, based on human body proportions: *"The planning of temples depends upon symmetry….. It arises from proportion … Proportion consists in taking a fixed module, in each case, both for the parts of a building and for the whole, by which the method of symmetry and proportion is put into practice. For without symmetry and proportion no temple can have a regular plan; that is, it must have an exact proportion worked out after the fashion of the members of a finely-shaped human body… Now the navel is naturally the exact centre of the body. For if man lies on his back with hands and feet outspread, and the centre of a circle is placed on his navel, his figure and toes will be touched by the circumference. Also a square will be found described within the figure, in the same way as a round figure is produced. For if we measure from the sole of the foot to the top of the head, and apply the measure to the outstretched hands, the breadth will be found equal to the height, just like sites which are squared by rule".*[4] On this passage is based Leonardo's Vitruvian man, synthesis of the harmonious and anthropocentric vision of the world during the Renaissance. Leonardo was able to represent in a singular illustration these three forms, the human figure, the square and the circle, relaying on the principle that the square and the circumference have different centres. The ideal proportions of the human body deriving from that figure correspond to the golden ratio of the square side to the circle radius. In this way geometry, thanks to the golden proportion, was able to connect technique and beauty. Leonardo's illustration proposes simultaneously the optic illusion of two transparent overlapping images reminding the dynamic evolution from a position to another.

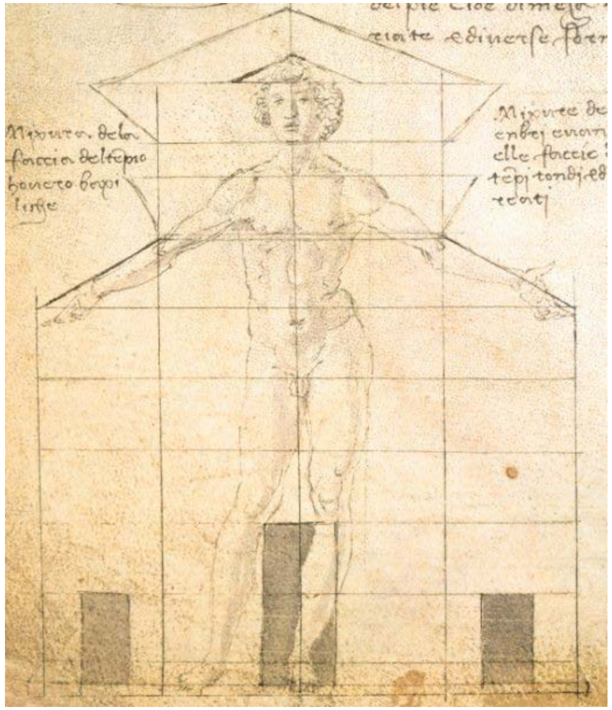
Total height = distance between the tips of the fingers of the two hands with open arms = 8 palms = 6 feet = 6 faces = 1,618 x height of the navel (distance between the ground and the navel).

Leonardo's Vitruvian Man was conceived according to the canon of human proportions that the famous Roman architect of the I century had postulated as introduction to his architectural theory. In the architecture of the Renaissance, in fact, Vitruvian theories were applied and were considered fundamental by several artists who were inspired to him. The Vitruvian principle of proportions, derived from those of human body, can be easily found in the drawings that complete the Renaissance treatises.

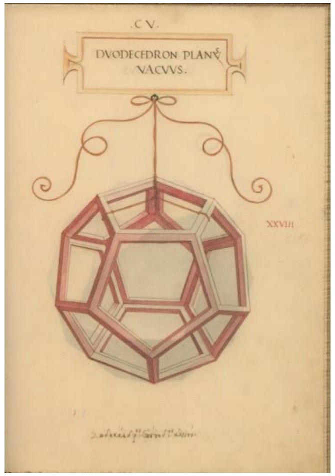
For example such an influence is present in the drawings by Francesco di Giorgio Martini, a Sienese architect that accomplished the Ducal Palace of Urbino commissioned by Federico da Montefeltro. There it is evident that the human figure if represented with the arms behind his back, determined the proportions of the plan of a church, as well as the human body, if represented with open arms, determines the height, the shape and the various parts of the facade. In the III book of his *Treatise of civil and militar architecture,* written at the court of Federico da Montefeltro in Urbino, in 1480, Francesco di Giorgio Martini deals with the theme of the form of the city developing a wide collection of urban schemes. All his cities have a radial plan, sometimes enriched with checkboard grid patterns somehow a heritage from ancient structures. With his design of the ideal city based on the proportions of the human figure, the architect confirms his classical formation, he asserts, in fact, that the square should be placed *"in the middle and in the very centre of it like the navel in the human body".*

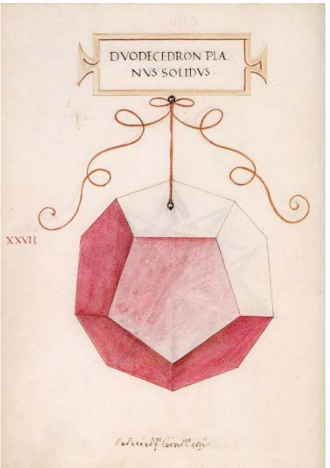


Francesco di Giorgio Martini, Proportioning plan of a sacred construction based on the human body, ca 1489 – 1501. Florence



Francesco di Giorgio Martini, Proportioning of the elevation of a sacred construction based on the human body, 1482 – 1486. Biblioteca Reale, Turin





Leonardo da Vinci, illustrations. The empty dodecahedron and the solid dodecahedron , for the edition of *De Divina Proporzione* by Luca Pacioli

Luca Pacioli and Leonardo da Vinci were the main responsible for the inclusion of the golden number in the orbit of beauty and art; during the Renaissance the golden ratio was the subject of particularly in-depth studies. In fact, in 1509, in Venice, Luca Pacioli published *De Divina Proporzione* (The Divine proportion) of 1498. In the text the proportions that we must follow to reach the perfection of beauty are asserted, exposed in the form of reflection on geometry. Actually the first artist to be interested in a particular way to geometry and to the "Platonic solids" was Piero della Francesca (1416-1492) who towards the end of his life replaced the artistic activity with the writing of several treatises about maths and perspective. The franciscan friar Luca Pacioli can be considered the divulgator in vernacular of the previous mathematical treatises and the work which had a deep impact on his contemporaries was the above mentioned *De divina proporzione*, illustrated by Leonardo da Vinci.

Leonardo in his drawings and manuscripts (collected in ten codes preserved in various European museums) also theorizes his beliefs about the art of painting, defending and supporting the fundamental connections between painting and mathematics. In his *Treatise on Painting*, written in 1498 , we can read: "*None, except a mathematician should read my works*." Today these illustrations, together with the *ideal man,* are considered authentic icons of a *forma mentis* that combines the artistic sensibility to the scientific one. Leonardo knows how to apply the scientific knowledge of human proportions to Pacioli's and Vitruvius's studies about beauty.

Leonardo has a dynamic vision of geometry, in his opinion solids are in a continuous transformation, rectangles change into squares as well as cubes into parallelepipeds and pyramids, moreover the rectilinear motion can become curved.

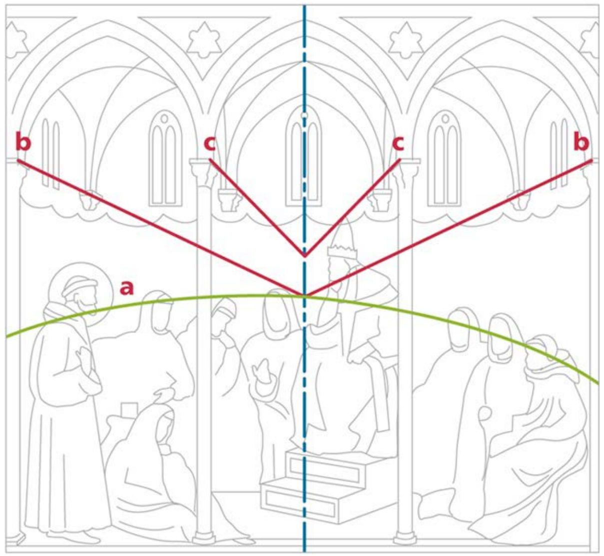
«Those who fall in love with practice without science are like a sailor who enters a ship without a helm or a compass, and who never can be certain whither he is going. Practice must always be founded on sound theory, and to this Perspective is the guide and the gateway , and without this nothing can be done well.»[5]

Leonardo da Vinci *«Treatise on Painting»*

5. Leonardo da Vinci , Treatise on Painting, http://www.discoveringdavinci.com/maxims-morals

The man in his real world experience has always used some features of visual perception. Those who work in three dimensional space are aware of the fact that the image received by the eye changes according to the position of the beholder and that the size of an object decreases with the increase of the distance.

For a long time the three-dimensional representations have been characterized by an empirical spirit, without scientific support and completely ignored in the Middle Ages, when the representation abandons the real world experience proposing religious images marked by a high symbolic content. In the fourteenth century, thanks to artists such as Duccio di Boninsegna (1255-1319), Giotto (1267-1336) and Ambrogio Lorenzetti (1285-1348) the artistic research ventures again to the conquest of space, the pictorial representation is no longer conceptual representation but vision of a real space, interpreted by the artist with chromatic tonality and environmental descriptions. For a plausible representation of reality, practical rules, such as the technique of foreshortened vision, the apparent convergence of parallel lines, the dimensioning of the figures on the basis of the distance, were elaborated, although still in an empirical way.



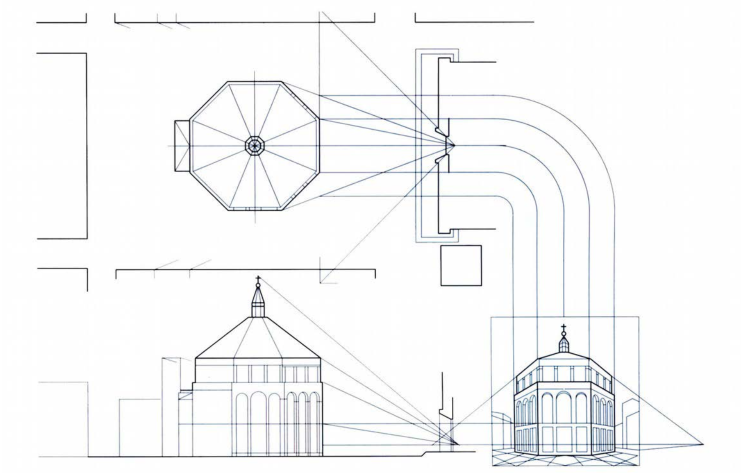
Giotto, Stories of Saint Francis, ca 1292 - 1296, fresco, Upper Church of the Basilica of San Francesco, Assisi

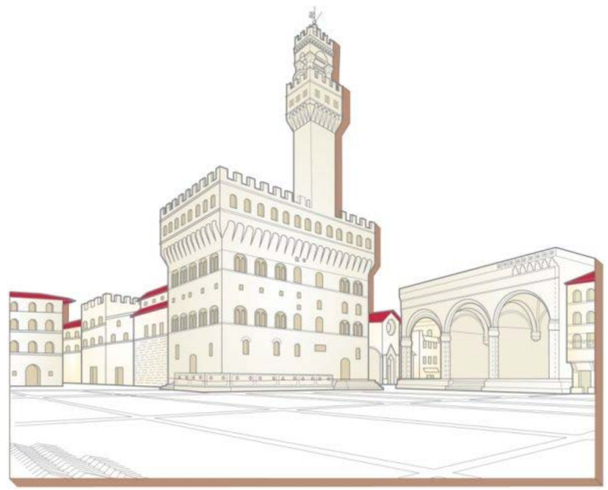
The Perspective

The perspective, fundamental innovation of the artistic research of the fifteenth century, enables to represent reality in a plausible way, as it highlights the depth of the real space and the relationship among all the elements that are inserted in it. In ancient times and throughout the Middle Ages, there is no differentiation between the *perspective* and the *optics* as both terms referred to the *science of vision*. The term perspective is used for the first time by Piero della Francesca in his treatise *De Prospectiva Pingendi (On Perspective for Painting,* 1475) , as derivation of the term *"perspectiva"* used to indicate Optics.

During the Renaissance, the term acquires the meaning of ***Science of representation,*** still in use today. It comes from the Latin *perspìcere* meaning "to see clearly" and this term exactly indicates one set of projections on a two-dimensional plane (i.e. characterized by two dimensions, length and height) of three-dimensional objects (characterized by three dimensions, length, height and depth).

The perspective graphic process realizes the representation of any object or of a set of objects on a sheet, so that the drawn image is as similar as possible to that really perceived. Filippo Brunelleschi, in 1413, discovered the geometrical rules of perspective representation and verified in practice the rules of perspective, using two tablets (prepared by himself but lost), built on an optical device. In the first (a front view of the Baptistery of San Giovanni) the geometric rules of central perspective were depicted, while In the second one (an angular view of Palazzo Vecchio) the rules of two-point perspective were used.



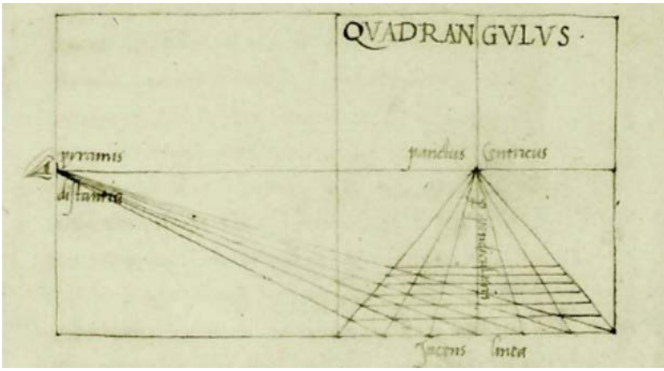


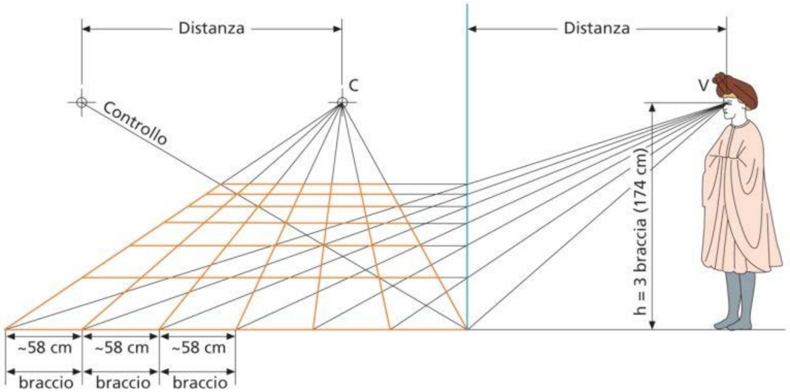
The Perspective

The Method of Leon Battista Alberti

The long lasting operations necessary for the implementation of a perspective through Brunelleschi's construction were greatly simplified and reduced in number by Leon Battista Alberti, the great humanist, painter and architect who was responsible for the process of perspective that became known as *costruzione abbreviata* (shortened building) exactly to underline the greater speed of execution.

Alberti in 1435 completed the first Latin treatise on perspective, *De Pittura* (On Painting) and translated it into vernacular the following year, dedicating it to Brunelleschi, recognizing, in this way, his priority in the discovery of perspective.





1- The base of the square is divided into parts, corresponding to the measure of the arm of a man (and equivalent to a third of his height). The height of the man on the picture plane is the level of the horizon.

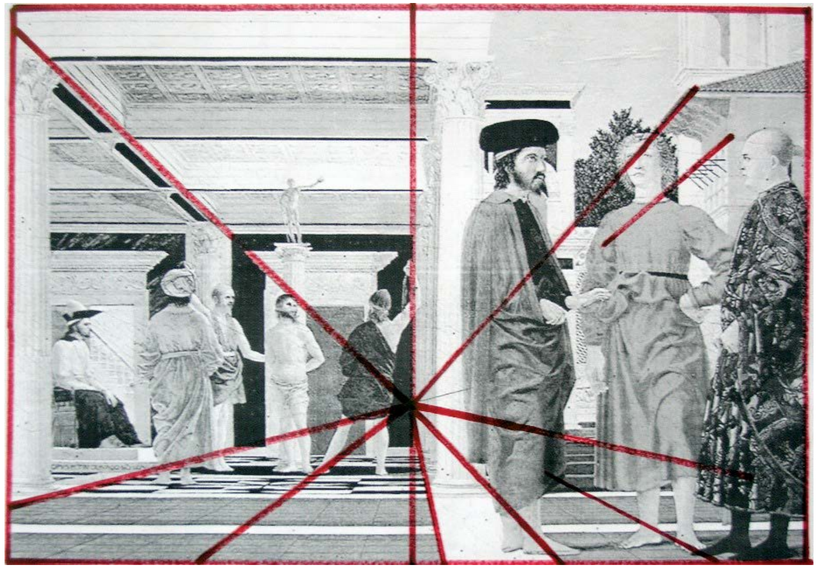
2 – The vanishing point or vanishing focus is identified and the points corresponding to the divisions in intervals of the base are converged to it , in order to trace the orthogonals.

3 – Laterally, the lines are drawn, starting from the divisions of an arm at the base of the painting, up to converge to the external distance point which constitutes the ideal distance of the observer from the painting. In this way the points of intersection on the base of the square are found.

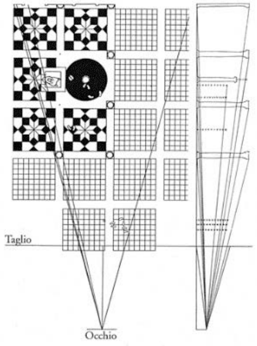
4 - On one of the vertical sides of the framework are reported the heights of the points of intersection drawing the horizontal divisions of the slabs of the floor.

Round about 1475, thanks to Piero della Francesca, the Renaissance had its first fully illustrated treatise on perspective: *De Prospectiva Pingendi* (On Perspective for painting) dedicated to Federico da Montefeltro. The term “pingendi” underlines that the perspective in which the artist is interested is a graphic matter.

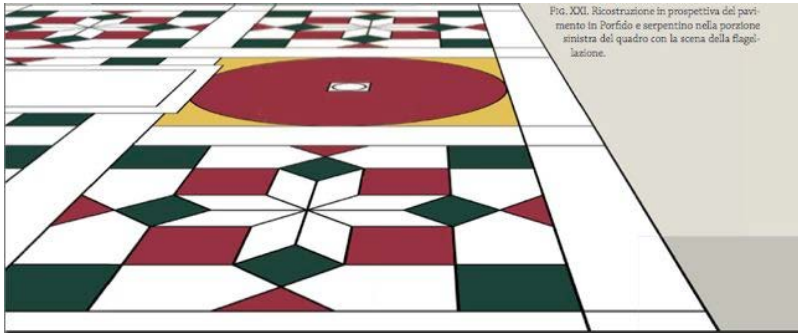
Piero della Francesca, *The Flagellation of Christ*, oil on panel, 59 x 81.5 cm, Galleria Nazionale delle Marche, Urbino



Perspective reconstruction of *The Flagellation of Christ*

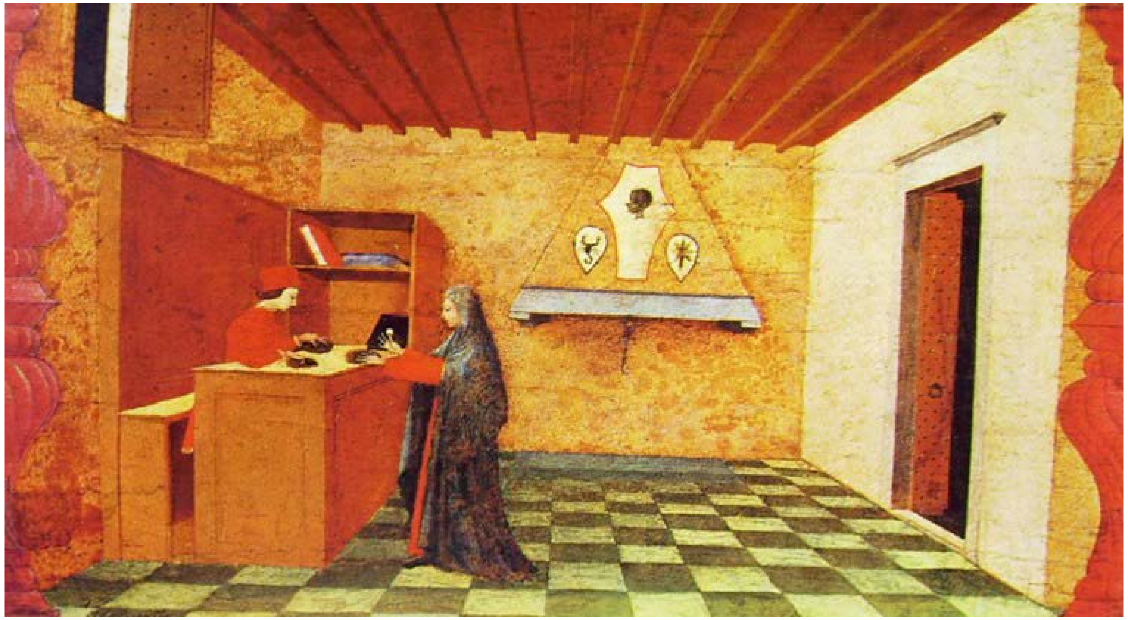


Reconstruction of the plan of *The Flagellation of Christ*



Perspective reconstruction of the paving of *The Flagellation of Christ*

Paolo di Dono, known as Paolo Uccello (Arezzo, 1397 - Florence, 1475) was a brilliant artist, fascinated by geometry and with a great imaginative freedom.
His entire path was characterized by the study of perspective and its possible variants, connecting the technique of the Renaissance to the persistences of the fourteenth century. The perspective is to Paolo Uccello a pure intellectual exercise.
The "*Miracle of the Desecrated host*" is a series of six autograph paintings by Paolo Uccello made with tempera technique on wood around 1465-69, all of them measure 33 x 58.5 cm. The pictorial complex, along with the altarpiece by Justus of Ghent (ca 1430 - 1480), is kept in the Galleria Nazionale delle Marche in Urbino.



Paolo Uccello, *The Miracle of the Desecrated Host*, 1465-1469, tempera on wood, Galleria Nazionale delle Marche, Urbino

In 1525 **Albrecht Durer** suggested a very "practical” definition of perspective: the artist observes his subject placing his eye in a particular position (point of view) and looks at it through a window (or frame) in which wires are pulled to form a grid. By reproducing each square of the grid, always viewed from the same point of view, a representation of the subject “in perspective" is obtained.

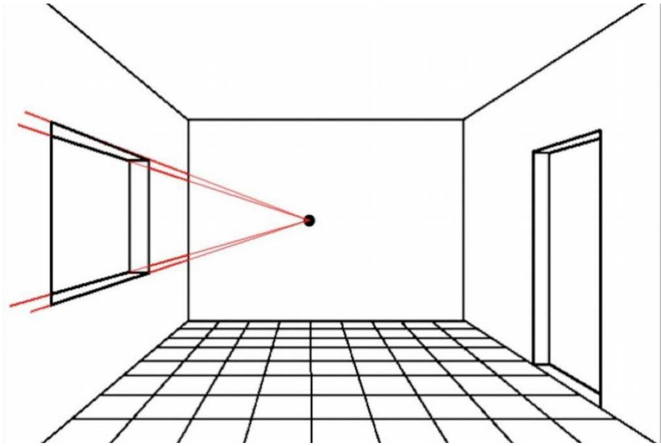
On the left it is possible to see the woman to be portrayed, while in the middle there is a frame in which the grid used to represent each single square is inserted .

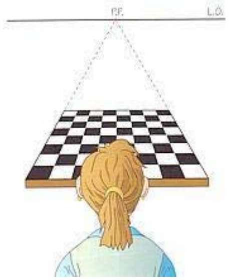
The artist, on the right, observes the subject from a precise point (the tip of the obelisk that is in front of his eye) and depicts the subject on the paper proposing the same "squares" that he sees in front of him.

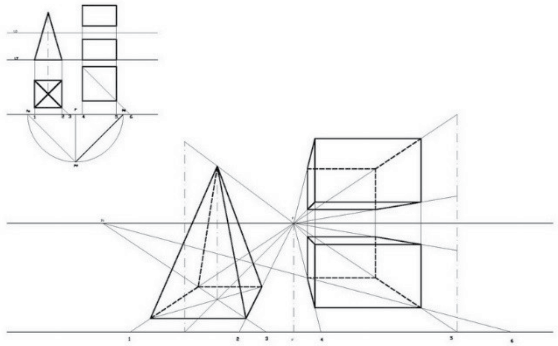
The perspective uses a monocular vision (concerning one eye) and not a stereoscopic vision (which is obtained from a binocular vision, that is through both eyes).

All the horizontal lines parallel to the painting and equidistant (which have the same distance) remain parallel, but their mutual distance decreases with the increasing of the distance from the painting.

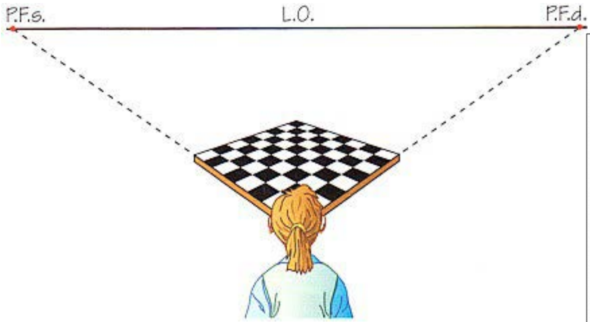
All the vertical lines parallel to the painting remain vertical, mutually parallel and if they lie on a plane parallel to the painting they maintain their reciprocal distances unchanged; on the other side, if they approach with progression towards their vanishing point their mutual distance decreases.

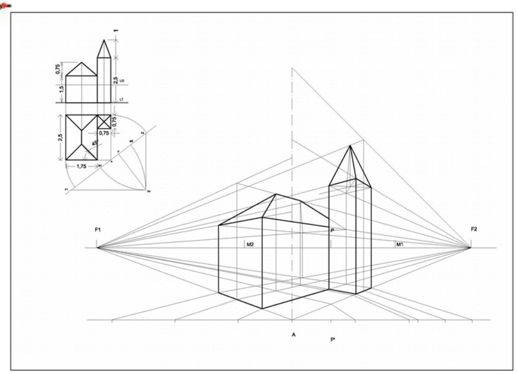






Central perspective





Two-point perspective

The *Sinopia* is a design realized before performing the fresco technique. The discovery of the existence of this representation took place after the Second World War, following the restoration campaigns, arranged for the recovery of the frescoes destroyed by bombing and fires.

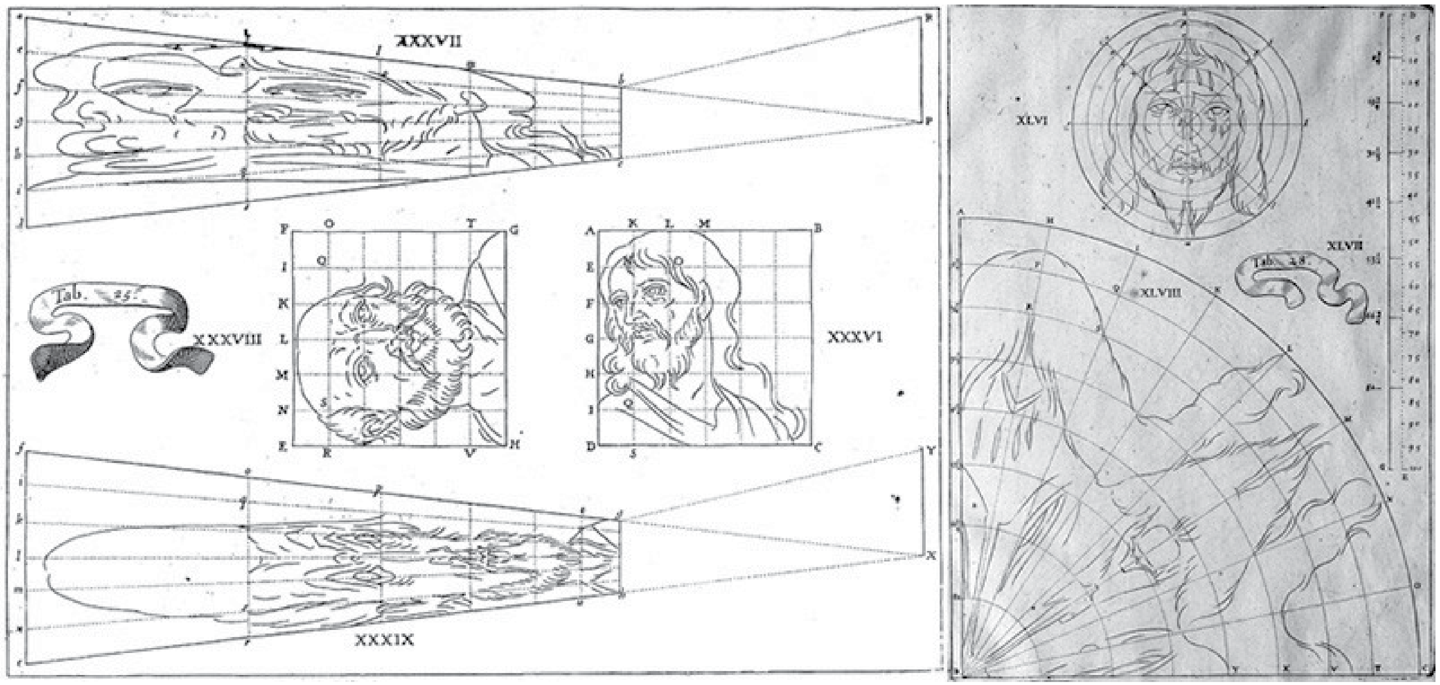
The “ tear” solution was adopted, an extreme choice but the only one possible given the conditions of the artworks terribly damaged, mostly, by fire. These proceedings have revealed all the processes that preceded the accomplishment of the works. The procedure consisted in gluing on the pictorial surface a thin cloth soaked in glue made from animal substances, once dried, the tearing of the canvas was realized , removing the painting from the plaster and discovering on the wall below the preparatory drawing. The pictorial layer, separated from its original support, was pasted onto a new base in order to be restored. The *sinopia* is made of clay from the city of Sinope in Pontus on the Black Sea (hence the Latin name *sinopis pontica*). With this clay at first the preparatory drawings were traced. The process is set at the beginning of the fifteenth century by Cennino Cennini who mentioned it in his "*Book of Art*", in which the phases of coat of the drawing and of the colour are coded.

In the XV century it is evident the gradual disappearance of the use of the *sinopia* as project tool. In order to better realize the perspective representations, changes are applied to the procedure: the *sinopia* is replaced by a preparatory carton, a design of the same dimensions of the fresco , which is pierced with metallic points along the contours of the figure and dusted with powdered charcoal or with the *sinopia*, so as to leave the draft of the design on the plaster. For large frescoes this procedure is done every day with different sections of the carton until the conclusion of the following drawing. The *sinopia* (preparatory design) is always performed by the master's hand, so it is in these drawings exactly that the distinctive style of the artist can be seen. Examples of these extraordinary preparatory drawings are in the Museo della Sinopia of Pisa.

The geometric greed is a tool that can be used to get different results.

For example a geometric grid drawn on a sketch can permit to enlarge the figure.

By varying the shape of the squares of a grid we can obtain not only an enlarged or reduced transposition of the original figure but also its deformation . The picture may become distorted because of the changes made to the reference system.

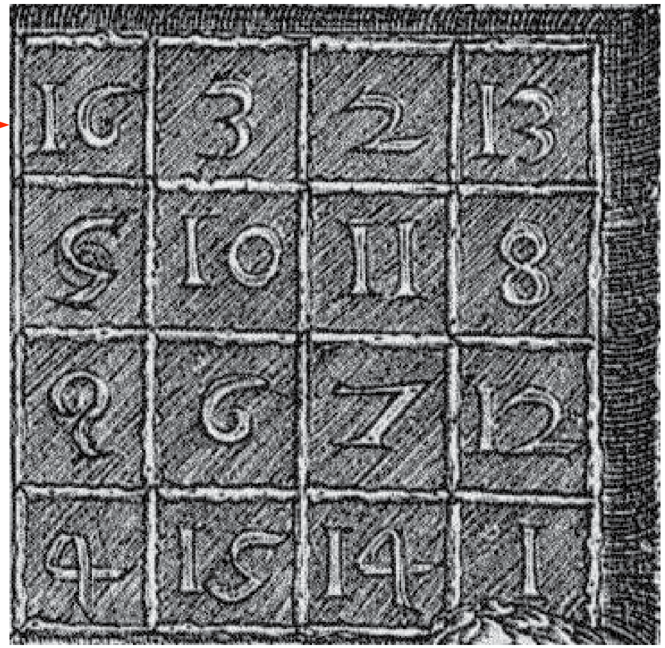


You can rationally deform an image .. and create an anamorphism of it.

An anamorphism is an optical illusion effect for which the image is projected in a distorted way, making the original subject recognizable just looking at the image from a precise position (anamorphosis: from the Greek , it consists of ana mórfosis = reconstructed form).

The magic square is an array of integers distinct in a square matrix such that the sum of the numbers in each row, in each column and in both diagonals always gives the same number; this integer is called magic constant or constant of magic , or magic sum of the square.

Dürer, *Melancolia*, 1514, engraving, 23,9 x 28,9 cm, Karlsruhe



Dürer, *The magic square* inserted in *Melancolia*

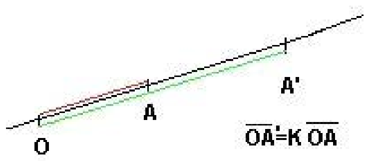
Dürer's work is very complex. In fact it is not just the sum of the numbers of horizontal, vertical and oblique rows to give 34 but also the sum of the numbers of the four sectors in which the square can be divided . Even the four numbers in the middle if summed up give 34 as well as the four numbers in the corners. Furthermore, taking a number at the corners and adding its opposite number 17 is obtained. Moreover, in the last row, the middle numbers form the number 1514, the year in which the work was made.

The Homothetia is a particular type of affine transformation. Let's see how it is defined.

Consider a point **O** in the plan and a real number **K** other than 0.The transformation **T** that to

each point **A** of the plan makes a correspondent **A'**, aligned with **O** and **A** such that: $\dfrac{\overline{OA'}}{\overline{OA}} = K$

is called Homothetia with centre **O** and ratio **K**.



The constant **K** is called Homothetia ratio:

- if **K** >0 the homothetia is defined direct,
- if **K** < 0 the homothetia is defined inverse.

O is the centre of homothetia.

We can apply the same transformation to more complex figures. In the following image

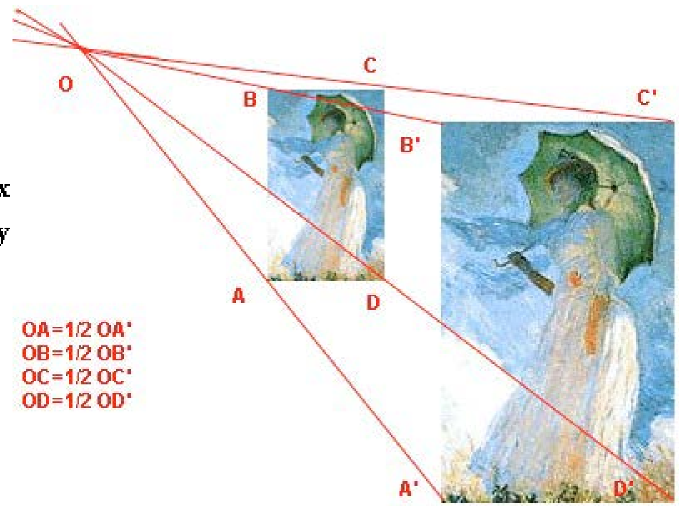
let's consider a homothetia of constant **K=1/2** we obtain a duplication of the original figure.

In case that the centre of homothetia

O corresponds with the origin of

axes is simple to give the analytic

equations of the homothetia $\begin{cases} X = Kx \\ Y = Ky \end{cases}$



Exercises

Multiples and Submultiples TABLE 1

Exercise 1

Establish without performing the division which numbers are multiples of 3 and 5 and give reasons for your reply:

a) 125

b) 111

c) 147

d) 1005

Exercise 2

1 km^2 of young forest produces about $2.5 \cdot 10^5$ kg of oxygen a year.

How much oxygen is produced in 1 m^2 of young forest?

Exercise 3

A bottle of mineral water which has a volume of 1.5L is used to fill a small children's pool with a volume of 3 m^3.

1. How many bottles are used?
2. To fill the bottle and empty it into the pool it takes 2 minutes. How long would it take to fill the pool?

Multiples and Submultiples TABLE 2

Exercise 4

A bicycle pump is essentially formed by a cylinder of diameter 2.0 cm and 30 cm long. A cyclist inflates a wheel pumping at a rate of 25 times per minute.

- What is the volume of the air pumped every time?
- What is the volume of the air pumped per second?
- Is the volume of the air pumped every second a unitary quantity?

Suppose that the air pumped into the wheel is compressed to half its original volume.

- What is the relationship between the density of the air in the pump before compression and the one in the wheel?

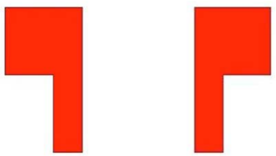
The Module TABLE 3

The repetition of a geometric figure according to rules defined by the artist creates a **geometric pattern** that is a composition characterized by new peculiarities; a rythm of constant or proportional development is acquired according to the norms which the pattern is based on.

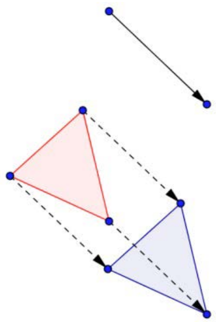
The original figure in the development of the pattern undergoes **simple transformations** or **composed transformations**. The repetition of an elementary base form, called *module,* can generate a bigger composite form.

The simple transformations are: translation, rotation, axial symmetry, symmetry, reflection.

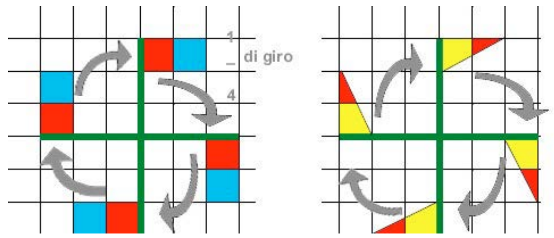
The composed transformations are obtained by summing the effects of two or more simple transformations.



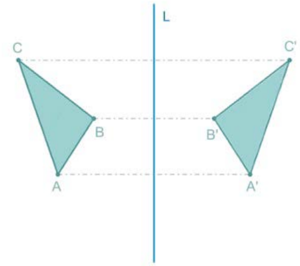
Reflection



Translation



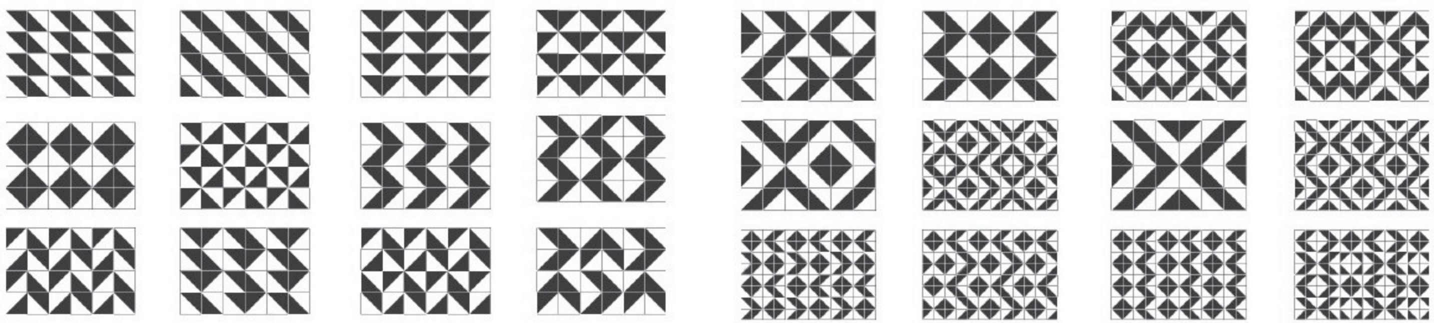
Rotation



Axial symmetry

Exercise 5

Using a simple geometric figure inserted in a grid create a modular composition respecting the simple transformations.



The Golden Section TABLE 4

Exercise 6

Construction of the pentagon given the side AB, by using the division of the segment in mean and extreme reason.

Exercise 7

Construction of the pentagon inscribed in a circumference, by using the division of the segment in mean and extreme reason

Exercise 8

Construction of the pentagram and of the pentagonal star

The Homothetia TABLE 5

Exercise 9

Now consider the following homothetia **T** of centre the origin of axes:

$$\begin{cases} X = -\dfrac{1}{2}x \\ Y = -\dfrac{1}{2}y \end{cases}$$

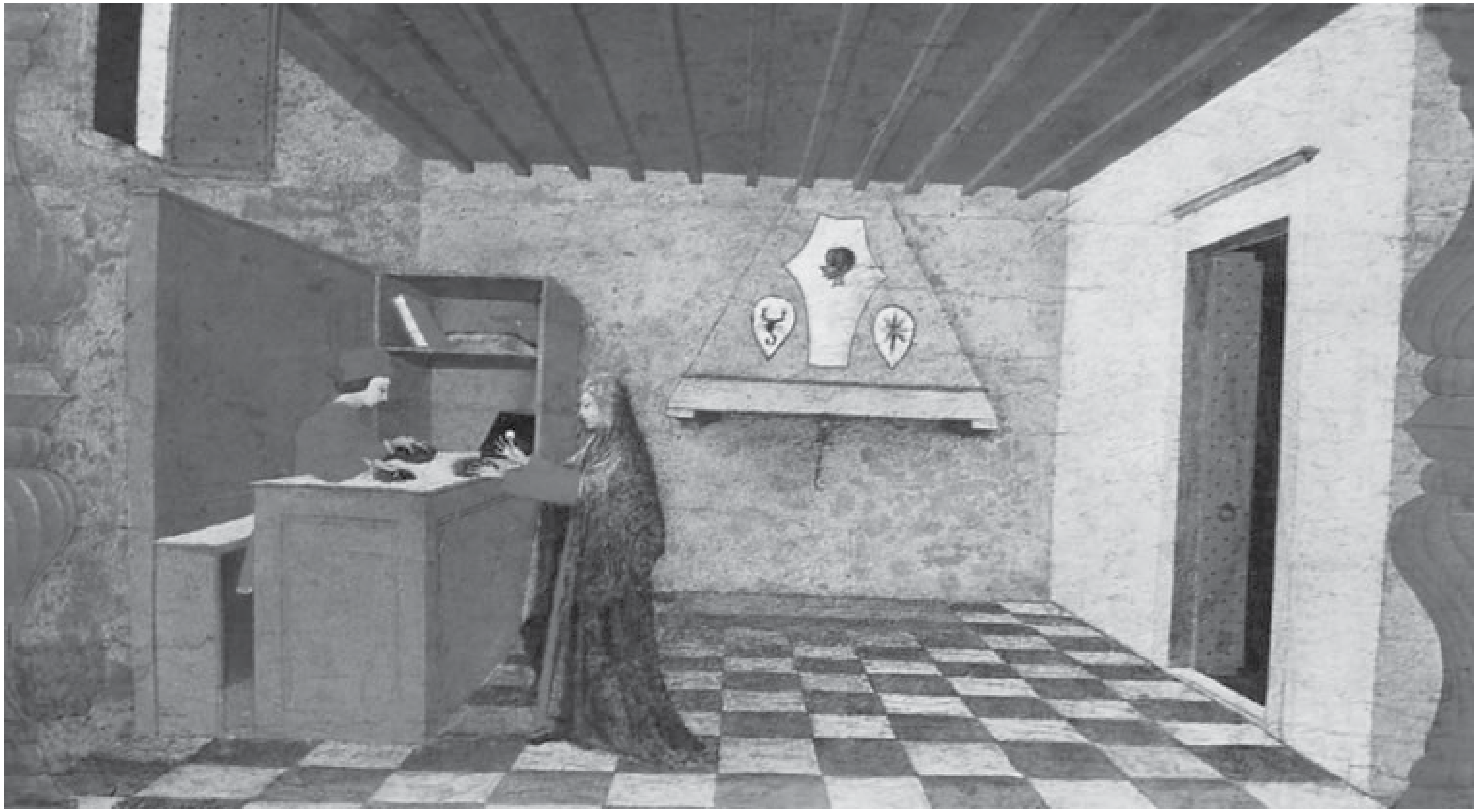
Draw in the Cartesian plane the circumference with *centre C* (1,0) and radius 1.

Determine the transformed figure . What relationship exist between the two circumferences ?

The Perspective

Exercise 10

With a colored pencil draw the perspective reconstruction of the artwork by Paolo Uccello



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